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ON THE CELLULARITY OF $\beta X - X$

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ABSTRACT. For a topological space X, let $c(X)$ denote the cellularity of X, and let $k(X)$ denote the least cardinal of a cobase for the compact subsets of X. It is shown that, if X is a completely regular Hausdorff space, $c(\beta X - X) \leqslant 2^{c(X)k(X)}$, and examples are given to show that this inequality is sharp. It is also shown that if X is an extremally disconnected completely regular Hausdorff space for which $c(\beta X - X) > 2^{k(X)}$, then $\beta X - X$ contains a discrete C^*-embedded subspace of cardinality $k(X)^+$.

1. **Preliminaries.** Our topological notation and terminology follows that of [6]. The Stone-Čech compactification of a completely regular Hausdorff space X is denoted by βX, and the cardinality of a set S is denoted by $|S|$. The discrete space of cardinality α is denoted by $D(\alpha)$. The cardinal 2^α is denoted by $\exp \alpha$.

Recall that the *cellularity* of a topological space Y, denoted by $c(Y)$, is defined by $c(Y) = \sup\{|\mathcal{G}|: \mathcal{G}$ is a family of pairwise disjoint nonempty open subsets of $Y\}$. A family $\mathcal{K}$ of compact subsets of Y is called a *cobase* for the compact subsets of Y, if every compact subset of Y is contained in a member of $\mathcal{K}$. We denote by $k(Y)$ the least cardinality of a cobase for the compact subsets of Y. The *Lindelöf number* of Y, denoted by $L(Y)$, is the least cardinal α such that every open cover of Y has a subcover of cardinality $\leqslant \alpha$.

The cardinal invariant $k(Y)$ is discussed by Arhangel′ skiĭ in [1]. Note that $k(Y) \geqslant L(Y)$ for each noncompact space Y.

Our principal reference for cardinal invariants of topological spaces is [8]. All hypothesized spaces in this paper are assumed to be completely regular and Hausdorff.

2. **The cellularity of** $\beta X - X$. Disjoint open subsets of $\beta X - X$ are considered in [3], where the following theorem is established (see 3.3 in [3]).

2.1. THEOREM (COMFORT-GORDON). *Let X be a completely regular space and let m be a cardinal number. Then the following assertions are equivalent*:

(i) *the space $\beta X - X$ admits a collection of m pairwise disjoint, nonempty open subsets*;

(ii) *the space X admits a collection $\mathcal{U}$ of cozero sets such that $|\mathcal{U}| = m$ and such that each member of $\mathcal{U}$ contains a noncompact zero-set, and, if U, V are distinct members of $\mathcal{U}$, then $U \cap V$ has compact closure in X.*

Our estimate for the cellularity of $\beta X - X$ uses the Comfort-Gordon result

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quoted above, together with the Erdös-Rado partition relation $(\exp\alpha)^+ \to (\alpha^+)^2_\alpha$ (see [5]).

2.2. THEOREM. *Let X be a completely regular Hausdorff space. Then $c(\beta X - X) \leqslant \exp(c(X)k(X))$.*

PROOF. Let $\alpha = c(X)k(X)$. If α is finite, the theorem is trivial, so we assume α is infinite. Let $\mathcal{K} = \{K_i: i < \alpha\}$ be a cobase of cardinality $\leqslant \alpha$ for the compact subsets of X. For the sake of contradiction, suppose $c(\beta X - X) > \exp\alpha$. Then there is a family of $(\exp\alpha)^+$ pairwise disjoint, nonempty open subsets of $\beta X - X$. (here, as is customary, we denote by m^+ the first cardinal exceeding m). By 2.1 above, there is a family $\mathcal{U}$ of cozero-sets in X such that

(i) $|\mathcal{U}| = (\exp\alpha)^+$,

(ii) each member of $\mathcal{U}$ contains a noncompact zero-set, and

(iii) if U, V are distinct members of $\mathcal{U}$, then $U \cap V$ has compact closure in X.

Let $[\mathcal{U}]^2$ denote the two-element subsets of $\mathcal{U}$. Let $A_0 = \{\{U, V\} \in [\mathcal{U}]^2: U \cap V \subseteq K_0\}$ and if $i < \alpha$ and A_j has been defined for each $j < i$, let $A_i = \{\{U, V\} \in [\mathcal{U}]^2: U \cap V \subseteq K_i\} - \bigcup_{j<i} A_j$. Since $\mathcal{K}$ is a cobase for the compact subsets, $[\mathcal{U}]^2 = \bigcup_{i<\alpha} A_i$. By the Erdös-Rado partition relation $(\exp\alpha)^+ \to (\alpha^+)^2_\alpha$ (see Theorem 39 of [5]), there is a subfamily $\mathcal{U}_1$ of $\mathcal{U}$ such that $|\mathcal{U}_1| = \alpha^+$, and an $i < \alpha$, such that $\{U, V\} \in A_i$ for all pairs U, V of members of $\mathcal{U}_1$. Now, let $\mathcal{G} = \{U - K_i: U \in \mathcal{U}_1\}$. Then $\mathcal{G}$ is a family of α^+ pairwise disjoint, nonempty open subsets of X. This is impossible, since $c(X) \leqslant \alpha$. This contradiction shows that $c(\beta X - X) \leqslant \exp\alpha$, proving 2.2.

The following proposition can be established in a straightforward manner. Its proof is left to the reader.

2.3. PROPOSITION. *Let X be locally compact, and not compact. Then $k(X) = L(X)$.*

2.2 and 2.3 yield the following corollary.

2.4. COROLLARY. *Let X be locally compact. Then $c(\beta X - X) \leqslant \exp(c(X)L(X))$. In particular, if X is locally compact and σ-compact, then $c(\beta X - X) \leqslant \exp(c(X))$.*

We now give examples to illustrate the sharpness of the inequality in 2.2.

2.5. EXAMPLES. (i) For each $n \in N$, let X_n be a copy of the one-point compactification of the discrete space of cardinality $\exp\exp\aleph_0$. Let X be the free union of the X_n. Clearly X admits a collection of $\exp\exp\aleph_0$ disjoint, countable, discrete, open-and-closed subsets. It follows that $c(\beta X - X) \geqslant \exp\exp\aleph_0$. However, since $k(X) = L(X) = \aleph_0$, we have $\exp(k(X)) < c(\beta X - X)$. This shows that the number $c(X)$ cannot be removed from the exponent in 2.2.

(ii) Let α be any cardinal number greater than or equal to $\exp\aleph_0$, and let X be the product of $\exp\alpha$ copies of the two-point discrete space $\{0, 1\}$. Write X as $\prod_{i<\exp\alpha} X_i$ where, for each i, $X_i = \{0, 1\}$.Let $\Sigma = \{x \in X: \{i < \exp\alpha: x_i \neq 0\}$ is countable$\}$. By Theorem 2 in [7], $\beta\Sigma = X$. Now for each $i < \exp\alpha$, let a_i be that point in X whose ith coordinate is 0 and whose jth is 1 for $j \neq i$. Then the subspace $S = \{a_i: i < \exp\alpha\}$ of X is homeomorphic to $D(\exp\alpha)$. Let

$Y = X - S$. Then $\Sigma \subseteq Y \subseteq X = \beta\Sigma$, so by 6.7 of [6], $\beta Y = X$ and $\beta Y - Y = S$. Now Y is a dense subset of X, so X and Y have the same cellularity. Since X is a product of separable spaces, $c(X) = \aleph_0$ (see Corollary 14 in [2]), and thus $c(Y) = \aleph_0$. Obviously $c(\beta Y - Y) = \exp\alpha$, and so $c(\beta Y - Y) > \exp(c(Y))$. This shows that the number $k(X)$ cannot be removed from the exponent in 2.2. It also shows that the gap between $c(Y)$ and $c(\beta Y - Y)$ can be made arbitrarily large. If one wants a locally compact example of the same phenomenon, let p denote the point of X all of whose coordinates are 1, let $T = S \cup \{p\}$, and put $U = X - T$. T is a copy of the one-point compactification of S, and hence U is locally compact. As above, $X = \beta U$, $c(U) = \aleph_0$, and $c(\beta U - U) = \exp\alpha$.

(iii) let X be the countable discrete space. Then $c(X) = k(X) = \aleph_0$ and $c(\beta X - X) = c$. This shows equality may be attained in 2.2.

(iv) Trivial examples show that the inequality in 2.2 can be strict. For example let p be a point of $\beta N - N$ and let $X = \beta N - \{p\}$. Then $c(\beta X - X) < 2^{c(X)k(X)}$.

Thus, in the sense described by the above examples, the inequality in 2.2 is best possible.

We conclude this note with the following result. Properties of extremally disconnected spaces used below may be found in 1H and 6M of [6].

2.6. THEOREM. *Let X be an extremally disconnected space. If $c(\beta X - X) > \exp(k(X))$ then $\beta X - X$ contains a discrete C^*-embedded subspace D of cardinality $k(X)^+$.*

PROOF. Arguing exactly as in the proof of 2.2, we find a compact subset K of X and a family $\mathcal{G}$ of $k(X)^+$ pairwise disjoint nonempty open subsets of X each of which is disjoint from K and none of which is relatively compact in X. Since βX is extremally disconnected, disjoint members of $\mathcal{G}$ have disjoint βX-closures. The open subspace $\beta X - K$ of βX is extremally disconnected, so its open subspace $T = \bigcup\{\mathrm{cl}_{\beta X}\, G - K\colon G \in \mathcal{G}\}$ is C^*-embedded in $\beta X - K$. For each $G \in \mathcal{G}$ let $p(G)$ denote a point in $\mathrm{cl}_{\beta X}\, G - X$. Then $\{p(G)\colon G \in \mathcal{G}\}$ is a discrete subspace D of $\beta X - X$ of cardinality $k(X)^+$. If $f \in C^*(D)$, define a function g from T to the real numbers by $g[\mathrm{cl}_{\beta X}\, G - K] = \{f(p(G))\}$. Then $g \in C^*(T)$, and can be continuously extended to $h \in C^*(\beta X - K)$. Then $h|\beta X - X$ is a continuous extension of f to $\beta X - X$. Hence D is C^*-embedded in $\beta X - X$.

2.7. COROLLARY. *If X is a locally compact extremally disconnected space and if $c(\beta X - X) > \exp(k(X))$, then $\beta X - X$ contains a copy of βD where D is the discrete space of cardinality $k(X)^+$.*

The above result is of some interest when contrasted with the theorem of Efimov [4, Theorem 8], that if K is a compact space of weight greater than $\exp\exp\exp\alpha$ and $c(K) \leqslant \alpha$, then K contains a copy of each extremally disconnected space of weight no larger than $(\exp\alpha)^+$, and thus contains a copy of $\beta D(\alpha)$ (and also a copy of $\beta D(\alpha^+)$ if we assume that $(\exp\alpha)^+ = \exp(\alpha^+)$). Efimov's result hypothesizes an upper bound on the cellularity of the space under question, while 2.7 above hypothesizes a lower bound on the cellularity of the space.

Finally, we note that $k(X)$ could be replaced in 2.6 and 2.7 by any cardinal greater than $k(X)$, and the results would remain valid.

The authors do not know whether 2.2 and 2.6 remain valid if $k(X)$ is replaced by $L(X)$.

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