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PRIMITIVE IDEALS IN GROUP RINGS OF POLYCYCLIC GROUPS

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ABSTRACT. If F is a field which is not algebraic over a finite field and G is a polycyclic group, then all primitive ideals of the group ring $F[G]$ are maximal if and only if G is nilpotent-by-finite.

We recall that a primitive ring is a ring with a faithful irreducible module. An ideal is primitive if the factor ring is a primitive ring.

If F is a field algebraic over a finite field and G a polycyclic group, then Roseblade has recently shown that every irreducible module for the group ring $F[G]$ is finite dimensional [5]. This implies that the primitive factor rings are simple Artin. On the other hand, if F is any other field and G is not abelian-by-finite, then Hall observed that $F[G]$ has infinite dimensional irreducible modules [2]. If G is finitely generated nilpotent, Zalesskiĭ proved that the primitive factor rings are at least simple for any field F [6]. In this paper we offer a converse to Zalesskiĭ's theorem by proving the

THEOREM. *If F is a field which is not algebraic over a finite field and G is a polycyclic group, then all primitive ideals of the group ring $F[G]$ are maximal if and only if G is nilpotent-by-finite.*

We warn the prospective reader of Zalesskiĭ's paper [6] that the word "primitive" has been translated as prime throughout.

LEMMA 1. *If G is polycyclic and H is a subgroup of finite index with all primitive ideals of $F[H]$ maximal, then all primitive ideals of $F[G]$ are also maximal.*

PROOF. The proof of Theorem 3 in [6] applies.

LEMMA 2. *Let G be polycyclic and H a subgroup of finite index in G. If $F[H]$ has a nonmaximal primitive ideal, then $F[G]$ does also.*

PROOF. By Lemma 1, we may assume H is normal in G. Let P be a primitive ideal of $F[H]$ properly contained in a maximal ideal Q. Let $1 = g_1, g_2, \ldots, g_n$ be a set of coset representatives for H in G. Let $\overline{P} = \bigcap_{i=1}^n g_i^{-1} P g_i$ and $\overline{Q} = \bigcap_{i=1}^n g_i^{-1} Q g_i$. $\overline{P} \subsetneqq \overline{Q}$ since equality would imply that $P \supseteq g_i Q g_i^{-1}$ for some i and hence P would be maximal. Let V be an irreducible $F[H]$ module with annihilator P. $\overline{V} = V \otimes_{F[H]} F[G]$ has finite length as an $F[H]$ module and hence as an $F[G]$ module. Let $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = \overline{V}$ be a $F[G]$ composition series for $\overline{V}$. The annihilator of $\overline{V}$ is $\overline{P}G$. $\overline{Q}G$ is a two-sided ideal

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and hence can be embedded in a maximal ideal M. Let $P_i = \operatorname{Ann}(W_i/W_{i-1})$. Since $P_n P_{n-1} \cdots P_1$ annihilates $\overline{V}$, $P_n P_{n-1} \cdots P_1 \subseteq \overline{P}G \subseteq \overline{Q}G \subseteq M$. Hence M contains P_i for some i. Now W_i/W_{i-1} contains a copy of Vg_j for some j and hence contains a copy of Vg_j for each j. Hence $P_i \cap F[H] = \operatorname{Ann}_{F[H]} W_i/W_{i-1} = \bigcap_{i=1}^n \operatorname{Ann}(Vg_i) = \overline{P}$. Since $M \cap F[H] \supseteq \overline{Q}G \cap F[H] = \overline{Q}$, we have $P_i \subsetneq M$ and P_i is a nonmaximal primitive ideal.

Proof of Theorem. If G is nilpotent-by-finite, the result follows from Zalesskiĭ [6] and Lemma 1. Conversely suppose G is not nilpotent-by-finite. Pick a subgroup K maximal among the subgroups with $N = \mathfrak{N}_G(K)$ of finite index and N/K not nilpotent-by-finite. Using Lemma 2, we may assume K is normal. Also since $F[G/K]$ is a homomorphic image of $F[G]$ we assume $K = 1$. The finite conjugate subgroup of G is trivial. Otherwise G has a nontrivial normal subgroup H whose centralizer C has finite index. $C/C \cap H \cong CH/H$ is nilpotent-by-finite by the maximality of K. Also $C \cap H$ is central in C and hence C and therefore G is nilpotent-by-finite, a contradiction. Let A be a maximal abelian normal subgroup. G contains a subgroup G_1 of finite index such that A contains a normal subgroup B of G_1 with the property that G_1 and all of its subgroups of finite index act rationally irreducibly on B [5, Lemma 2]. By Lemma 2, we assume $G = G_1$. A is torsion free since G has trivial finite conjugate subgroup. The rank of B is at least two for the same reason. We may clearly replace B with $QB \cap A$. QA is a $Q[G/A]$ module and QB is an irreducible submodule. We claim that QA is an essential extension of QB. Suppose to the contrary that T is a nontrivial $Q[G/A]$ submodule with $T \cap QB = 0$. $T \cap A$ is a nontrivial normal subgroup of G and $G/(T \cap A)$ is nilpotent-by-finite. This is impossible since $G/T \cap A$ contains an isomorphic copy of B and the rank of B is at least two. Again using Lemma 2, we may assume G/B is nilpotent. Let U/QB be an irreducible $Q[G/A]$ submodule of QA/QB. $U \cap A/B$ must intersect the center of G/B nontrivially. Therefore, U/QB must have Q dimension 1. Clearly $\operatorname{Ann}(U) \subseteq \operatorname{Ann}(QB)$. If $P_1 = \operatorname{Ann} U/QB$ and $P_2 = \operatorname{Ann} QB$, then $P_1 \neq P_2$ since $Q[G/A]/P_1 \cong Q$ and $Q[G/A]/P_2$ has dimension greater than 1. If $\operatorname{Ann} U = \operatorname{Ann} QB$, then $P_2 \subseteq P_1$, but this is impossible since each has cofinite dimension and hence are maximal. Therefore, $\operatorname{Ann}(U) \subsetneq \operatorname{Ann}(QB)$. By [4], there is an $x \neq 0$ in the center of $Q[G/A]/\operatorname{Ann}(U)$ with x in $\operatorname{Ann}(QB)/\operatorname{Ann}(U)$. x induces an isomorphism of U/QB onto QB. But this is impossible. Therefore, we have $QA = QB$ and hence $A = B$. We now show $C(B) = B$. If not, take x in $C(B) - B$ with xB in $Z(G/B) \cap C(B)/B$. Then $D = \langle x, B\rangle$ is an abelian normal subgroup contradicting the maximality of $A = B$. Since F is not algebraic over a finite field, there exists a monomorphism of B into the multiplicative group of F. This defines an $F[B]$ module structure on F. We denote this module by V. Let $P = \operatorname{Ann}_{F[B]} V$. $\overline{V} = V \otimes_{F[B]} F[G]$ is an irreducible $F[G]$ module. This follows since if g_1 and g_2 are in different cosets of B, then Vg_1 and Vg_2 are not isomorphic as $F[B]$ modules since $C[B] = B$. This implies $g_i P g_1^{-1} \neq g_2 P g_2^{-1}$. Annihilator of $\overline{V}$ is $(\bigcap_{g \text{ in } G} gPg^{-1})G$. But this is zero by Bergman's theorem [1]. Hence $\operatorname{Ann}(\overline{V})$ is a nonmaximal primitive ideal.

The proof can be simplified considerably if F is a large field. More specifically, if the transcendence degree F is larger than the rank of G, a

theorem of Passman may be applied to produce lots of primitive ideals easily [3].

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