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THE COMMUTATOR SUBGROUP OF A FREE TOPOLOGICAL GROUP NEED NOT BE PROJECTIVE

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ABSTRACT. It is shown that the commutator subgroup of the free topological group on the n-sphere is, for $n > 1$, not projective and hence not free topological. The proof depends on the computation of the mod 2 cohomology ring of the classifying space of the commutator subgroup.

1. **Introduction.** The free topological group FX on a pointed space X was defined by Graev [3]. A subgroup of a free topological group need not be free topological, [3], [5], while [2] gives examples of subgroups of free topological groups which are not projective. The examples in [2] are not connected. We shall prove

THEOREM (1.1). *Let FS^n be the (Graev) free topological group on the n-sphere, and CS^n its commutator subgroup. For $n > 1$, CS^n is not projective, and hence not free topological.*

This gives examples of nonprojective subgroups which are closed and path-connected.

The methods are homotopy theoretic. It is not known whether CS^1 is free; it is certainly isomorphic to a closed subgroup of FS^2 for which the inclusion is a homotopy equivalence, so homotopy theory cannot help further here.

I would like to thank R. Brown for raising the question.

2. **Proof of Theorem (1.1).**

DEFINITION (2.1). The topological group G is *projective* if given any diagram of morphisms of topological groups

$$\begin{array}{ccc} & & G \\ & & \downarrow \psi \\ H & \xrightarrow{f} & K \end{array}$$

such that f has a continuous section, then ψ lifts to a morphism $G \to H$.

Clearly a free topological group is projective.

PROPOSITION (2.2). *Let G be a projective topological group which has the based homotopy type of a CW-complex and BG its classifying space. For any multiplicative cohomology theory h^*, all products in the ring $\tilde{h}^*(BG)$ vanish.*

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PROOF. There are continuous homomorphisms $G \xrightarrow{s} FG \xrightarrow{f} G$ such that $fs = 1_G$, where f extends the identity $G \to G$, and the existence of s follows from the projectivity of G. This implies that BG is a retract of BFG. Since G has the homotopy type of a CW-complex, BFG has the homotopy type of the suspension SG [4], [7]. The result follows; see, for example, Proposition 13.66 of [9].

If X is the realisation of a simplicial set, as we may assume for $X = S^n$, FX and CX may be constructed simplicially; see [2]. If AX denotes the free abelian group on X, there is an exact sequence $CX \to FX \to AX$.

It follows from the simplicial constructions that, up to homotopy, there is a fibration $AX \to BCX \to BFX$, to which we can apply the Serre spectral sequence; see [6, Chapter VI].

We will prove Theorem (1.1) by showing

PROPOSITION (2.3). *The* mod 2 *cohomology ring of* BCS^n, *for* $n > 1$, *has a nontrivial multiplicative structure.*

PROOF. AS^n is a $K(\mathbf{Z}, n)$, [1, Theorem 5.12], or [6, Theorem 24.5], and BFS^n has the homotopy type of S^{n+1}.

The Serre spectral sequence

$$H^*(S^{n+1}; H^*(\mathbf{Z}, n; \mathbf{Z}/2)) \Rightarrow H^*(BCS^n; \mathbf{Z}/2)$$

has d_{n+1} as its only possible nonvanishing differential.

Let $x \in H^n(\mathbf{Z}, n; \mathbf{Z}/2) = E_2^{0,n}$ and $y \in H^{n+1}(S^{n+1}; \mathbf{Z}/2) = E_2^{n+1,0}$ be generators.

The homotopy exact sequence of the fibration shows that BCS^n is $(n+1)$-connected, thus $d^{n+1}x = y$.

Recall that $H^*(\mathbf{Z}, n; \mathbf{Z}/2)$ is, for $n > 1$, a polynomial algebra on the elements $\mathrm{Sq}^I x$, where I ranges over all admissible sequences $(i_1, \ldots, i_r)$ of excess $< n$ with $i_r \geqslant 2$ [8]. Since x is transgressive so are all the $\mathrm{Sq}^I x$. It is then easy to see that the E_∞-term of the spectral sequence is the tensor product of the exterior algebra on xy with the polynomial algebra on the elements x^2 and $\mathrm{Sq}^I x$, for $I \neq (0)$. Hence the proposition is proved.

For $n = 1$ the situation is quite different. $AS^1 \to BCS^1 \to BFS^1$ is, up to homotopy, the Hopf S^1-bundle over S^2.

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