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COMBINATORIAL FUNCTIONS AND INDECOMPOSABLE CARDINALS

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ABSTRACT. Combinatorial functions are used to replace indecomposable cardinals in certain types of set theoretic arguments. This allows us to extend decidability results from nonchoice set theories to set theories with a principle of linear ordering.

1. **Introduction.** Let ZF be Zermelo-Fraenkel set theory and let ZFO be ZF plus an axiom asserting that every set can be linearly ordered. ω = the nonnegative integers, Ω = the cardinal numbers, and Δ = the Dedekind cardinals $= \{x \in \Omega | x \neq x + 1\}$. Consider a first order language L containing individual variables $u_0, u_1, \ldots, v_0, v_1, \ldots$, a binary functor $+$, and a symbol $=$ denoting equality. L is interpreted in ω or Δ by letting $+$ denote cardinal addition. Because ω and Δ are commutative semigroups we take the liberty of putting terms of L in the normal form $\sum_{i<n} a_i u_i$ where $\sum$ denotes summation, $a_i \in \omega$, and $a_i u_i$ is the term consisting of u_i summed with itself a_i times. If φ is a sentence of L we write $\omega \vDash \varphi$ if φ is true in ω, and by $Q \vdash [\Delta \vDash \varphi]$ we mean that it is a theorem of Q that φ is true in Δ.

An *AE special Horn sentence* is a sentence of L having the form

$$(\forall u_0, \ldots, u_{m-1})(\alpha \to (\exists v_0, \ldots, v_{n-1})\beta) \tag{1}$$

where $\alpha(u_0, \ldots, u_{m-1})$ has the form

$$\bigwedge_{j<q} \sum_{k<m} a_{jk} u_k = \sum_{k<m} a'_{jk} u_k \tag{2}$$

and $\beta(u_0, \ldots, u_{m-1}, v_0, \ldots, v_{n-1})$ has the form

$$\bigwedge_{j<r} \sum_{k<m} b_{jk} u_k + \sum_{k<n} c_{jk} v_k = \sum_{k<m} b'_{jk} u_k + \sum_{k<n} c'_{jk} v_k . \tag{3}$$

In [2] it is mentioned in passing that

PROPOSITION A. *If φ is an AE special Horn sentence and $\omega \vDash \varphi$ then $ZF \vdash [\Delta \vDash \varphi]$.*

In this paper we describe two very different ways to obtain a converse of this result. An *AE special sentence* is a sentence of L having the form

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$$(4)\qquad (\forall u_0, \ldots, u_{m-1})\Big(\alpha \to (\exists v_0, \ldots, v_{n-1}) \bigvee_{i<p} \beta_i\Big)$$

where α has the form (2) and each β_i has the form (3). A *Horn reduct* of this sentence is any one of the Horn sentences

$$(5)\qquad (\forall u_0, \ldots, u_{m-1})(\alpha \to (\exists v_0, \ldots, v_{n-1})\beta_i).$$

From Proposition A it readily follows that if φ is an AE special sentence having a Horn reduct ψ such that $\omega \vDash \psi$ then ZF $\vdash [\Delta \vDash \varphi]$. Conversely

THEOREM 1. *If ZF is consistent and φ is an AE special sentence such that ZF $\vdash [\Delta \vDash \varphi]$ then $\omega \vDash \psi$ for some Horn reduct ψ of φ.*

THEOREM 2. *If ZF is consistent and φ is an AE special sentence such that ZFO $\vdash [\Delta \vDash \varphi]$ then $\omega \vDash \psi$ for some Horn reduct ψ of φ.*

Theorem 2 of course implies Theorem 1. We have stated them separately because we offer two different proofs. Theorem 1 is proved using indecomposable cardinals and our proof is similar to others which appear in the literature for various structures (cf. [1], [3], [5]). In ZFO on the other hand every infinite cardinal is decomposable and this prevents us from proving Theorem 2 in the same way as we proved Theorem 1. Theorem 2 is proved using combinatorial functions, and the details of the argument once again suggest the intimate connection between this class of functions and cardinal arithmetic in ZFO.

By combining Proposition A with either Theorems 1 or 2 we obtain the following decision results. First, that $\{\varphi \in L | \varphi$ is an AE special sentence and ZF $\vdash [\Delta \vDash \varphi]\}$ is recursive, and second, that $\{\varphi \in L | \varphi$ is an AE special sentence and ZFO $\vdash [\Delta \vDash \varphi]\}$ is recursive. In [2] it is shown that $\{\varphi \in L |$ ZF $\vdash [\Delta \vDash \varphi]\}$ is not recursive. Whether this also holds with ZF replaced by ZFO is still an open problem. However we do conjecture a positive solution to the decision problem of the additive theory of Δ^*, the group formed by taking differences of members of Δ, in either ZF or ZFO. We have not been able to solve this problem despite thinking about it for some time. However we do have some partial results which are interesting in terms of replacing arguments involving indecomposable cardinals by ones involving combinatorial functions.

Besides being able to prove in ZF that Δ is a commutative semigroup (with respect to addition and 0 as the identity), we can also prove using Proposition A that Δ satisfies the universal closures of

$$(6)\qquad x + z = y + z \to x = y \quad \text{(cf. [8])},$$

$$(7)\qquad nx = ny \to x = y, \quad \text{for } 0 < n < \omega \text{ (cf. [8])},$$

where nx has its usual inductive definition. From (6) and (7) we can easily show that Δ can be extended to a torsion free Abelian difference group (TFAG) which we call Δ^* (= *the Dedekind integers*, a typical member having the form $x - y$ where $x, y \in \Delta$). One possible way to show that the theory of a TFAG is decidable is to give a complete set of axioms for it which is recursive. Such a method was devised in [7]. Let $(G, +)$ be a TFAG, $m, n \in \omega$, and $x_0, \ldots, x_{m-1} \in G$. $x_0, \ldots, x_{m-1}$ are said to be *strongly line-*

arly independent (mod n) if for each sequence $a_0, \ldots, a_{m-1} \in \omega$ and $y \in G$, $\sum_{i<m} a_i x_i = ny$ implies that each a_i is congruent to 0 $(\mathrm{mod}\ n)$ in the ordinary arithmetical sense. Let ψ_{mn} be a group theoretic sentence asserting that there exist m elements strongly linearly independent (mod n). The principal result of [7] when applied to TFAG's is

PROPOSITION B. *An extension of the theory of TFAG's is complete if and only if it is consistent and contains for any two numbers $m > 0$, $n > 1$ either the sentence ψ_{mn} or its negation.*

In [6] it is shown that a complete axiomatization of the group of isolic integers can be obtained by adding to the axioms of a TFAG each ψ_{mn}. This result is obtained by using indecomposable isols. Applying the methods of [6] to cardinals we get

THEOREM 3. *If ZF is consistent then so is ZF + $\{\Delta^* \models \psi_{mn} | m > 0, n > 1\}$.*

We have known for some time that Theorem 3 could be extended to ZFO. However we were not satisfied with our proof because it involved using objects in a certain model of ZFO which had an ad hoc character. Now however, using combinatorial functions we obtain

THEOREM 4. *If ZF is consistent then so is ZFO + $\{\Delta^* \models \psi_{mn} | m > 0, n > 1\}$.*

Thus again we have given an instance where a theorem about ZF is extended to ZFO by replacing a proof using indecomposable cardinals by one using combinatorial functions. It is to be hoped that these examples will lead to a general metatheory about cardinal arithmetic in ZF and in ZFO.

2. **Argument.**

LEMMA 1 ([1], [3]). *If $x, y \in \omega$ then $\sim (x = 1 \wedge y = 0)$ if and only if $(\exists z \in \omega)(2z \leqslant x + y \wedge x \leqslant 3z)$.*

PROOF. Assume $\sim (x = 1 \wedge y = 0)$. If x is even, take $z = x/2$. Clearly $2z = x \leqslant x + y$ and $x \leqslant 3x/2$. If $x \neq 1$ and x is odd, take $z = (x-1)/2$. Then $2z = x - 1 \leqslant x + y$. If $3z < x$ then $(3x/2) - (1/2) < x$ so $x < 1$ and x is even. If $x = 1$ and $y \neq 0$ take $z = 1$. Then $2z = 2 \leqslant x + y$ and $x = 1 \leqslant 3 = 3z$. This proves the left to right implication. Conversely assume $x = 1 \wedge y = 0$. Then $2z \leqslant x + y$ implies $z = 0$ so that $x \nleqslant 3z$. Q.E.D.

COROLLARY. *If $p \geqslant 1$ and $x_j \in \omega$ for $j < p$ then $\sim (x_i = 1 \wedge \bigwedge_{i \neq j < p} x_j = 0)$ if and only if $(\exists z \in \omega)(2z \leqslant \sum_{j<p} x_j \wedge x_i \leqslant 3z)$ for each $i < p$.*

Let φ be an AE special sentence of the form (4) and let θ_p be

$$(8) \qquad (\forall u_0, \ldots, u_{p-1}) \bigvee_{i<p} (\exists v)\Big(2v \leqslant \sum_{j<p} u_j \wedge u_i \leqslant 3v\Big).$$

For the following lemma let Q be either ZF or ZFO.

LEMMA 2. *If φ is an AE special sentence such that $Q \vdash [\Delta \models \varphi]$, but no Horn reduct of φ is true in ω then $Q \vdash [\Delta \models \theta_p]$.*

PROOF. Since no Horn reduct of φ is true in ω, for each $i < p$ there are $x_{0i}, \ldots, x_{(m-1)i} \in \omega$ such that

$$\omega \models \alpha(x_{0i}, \ldots, x_{(m-1)i}), \tag{9}$$

$$\omega \models (\forall v_0, \ldots, v_{n-1}) \sim \beta_i(x_{0i}, \ldots, x_{(m-1)i}, v_0, \ldots, v_{n-1}). \tag{10}$$

Let $\alpha'(u_0, \ldots, u_{p-1})$ be

$$\alpha\left(\sum_{i<p} x_{0i} u_i, \ldots, \sum_{i<p} x_{(m-1)i} u_i\right)$$

and let $\beta_i'(u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})$ be

$$\beta_i\left(\sum_{i<p} x_{0i} u_i, \ldots, \sum_{i<p} x_{(m-1)i} u_i, v_0, \ldots, v_{n-1}\right).$$

Since $\alpha(u_0, \ldots, u_{m-1})$ is a conjunction of homogeneous linear equations and any linear combination of solutions of such a system is itself a solution, (9) implies

$$\omega \models (\forall u_0, \ldots, u_{p-1})\alpha'. \tag{11}$$

Also it follows from (10) that

$$\omega \models (\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})\left(\beta_i' \rightarrow \sim \left(u_i = 1 \wedge \bigwedge_{i \neq j < p} u_j = 0\right)\right) \tag{12}$$

for each $i < p$. Then applying the corollary we get

$$\omega \models (\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1})\left(\beta_i' \rightarrow (\exists v)\left(2v \leqslant \sum_{j<p} u_j \wedge u_i \leqslant 3v\right)\right) \tag{13}$$

for each $i < p$. Proposition A applies directly to (11) and by eliminating $u_0 \leqslant u_1$ with $(\exists v) u_0 + v = u_1$ we may also apply it to (3). Thus

$$Q \vdash [\Delta \models (\forall u_0, \ldots, u_{p-1})\alpha'], \tag{14}$$

$$Q \vdash \left[\Delta \models (\forall u_0, \ldots, u_{p-1}, v_0, \ldots, v_{n-1}) \left(\beta_i' \rightarrow (\exists v)\left(2v \leqslant \sum_{j<p} u_j \wedge u_i \leqslant 3v\right)\right)\right] \tag{15}$$

for each $i < p$. Now by hypothesis $Q \vdash [\Delta \models \varphi]$ so by recalling the definitions of α', β_i' we obtain

$$Q \vdash \left[\Delta \models (\forall u_0, \ldots, u_{p-1})\left(\alpha' \rightarrow (\exists v_0, \ldots, v_{n-1}) \bigvee_{i<p} \beta_i'\right)\right]. \tag{16}$$

Combining (14), (15) and (16) gives $Q \vdash [\Delta \models \theta_p]$. Q.E.D.

Our proofs of Theorems 1 and 2 are now easily seen to consist of showing that $\Delta \models \theta_p$ cannot be proved in Q if $p \geqslant 1$. To this end we introduce these notions. Let x, y be cardinals. x is *indecomposable* if $(\forall y, z \in \Omega)(x = y + z$

$\rightarrow (y \in \omega \vee z \in \omega))$. x and y are *comparable* if $x \leqslant y$ or $y \leqslant x$. It is not hard to show that indecomposable cardinals are all Dedekind (either finite or infinite).

LEMMA 3. *Assume $p \geqslant 1$ and $x_0, \ldots, x_{p-1}$ are pairwise incomparable indecomposable cardinals. In ZF we can prove that if $2y \leqslant \sum_{j<p} x_j$ then $y \in \omega$.*

PROOF. If $2y \leqslant \sum_{j<p} x_j$ then the refinement property (cf. [8]) gives cardinals $z_{jk}, j < p, k < 2$ such that $\sum_{j<p} z_{j0} = y = \sum_{j<p} z_{j1}$ and $z_{i0} + z_{i1} \leqslant x_i$ for each $i < p$. If y is infinite then z_{i0} is infinite for some $i < p$. Since the x_j are indecomposable, $z_{i1} \leqslant x_i - z_{i0}$ is finite. Thus $x_i \leqslant q + \sum_{i \neq j<p} x_j$ for some $q \in \omega$. Another application of refinement gives cardinals $z_j, i \neq j < p$ and $q' \in \omega$ such that $x_i = q' + \sum_{i \neq j<p} z_j$, $q' \leqslant q$ and $z_j \leqslant x_j$. Exactly one z_j is infinite and $x_j - z_j$ is finite. Thus for some $q'' \in \omega$ and $j \neq i$ we have $x_i \leqslant q'' + x_j$. But then x_i and x_j are comparable. Contradiction. Q.E.D.

PROOF OF THEOREM 1. It is well known (cf. [1], [2]) that if ZF is consistent then it has a model M containing countably many pairwise incomparable indecomposable cardinals. Assume φ is an AE special sentence such that ZF $\vdash [\Delta \models \varphi]$, but that no Horn reduct of φ is true in ω. Then by Lemma 2 there is a $p \geqslant 1$ such that $\Delta \models \theta_p$ is true in M. Choose $x_0, \ldots, x_{p-1} \in M$ which in the sense of M are pairwise incomparable indecomposable cardinals. But then in M, (8) gives a $y \in M$ such that $2y \leqslant \sum_{j<p} x_j$ and $x_i \leqslant 3y$ for some $i < p$. The latter condition implies that y is infinite while Lemma 3 implies that y is finite. Contradiction. Q.E.D.

To prove Theorem 2 we need the following:

PROPOSITION C ([3]). *If $(Q)\varphi$ is a sentence of L, where Q is a string of quantifiers and φ is a quantifier free positive matrix, and ZFO $\vdash [\Delta \models (Q)\varphi]$ then ZF$\vdash$ there is a Horn reduct φ' of φ and almost combinatorial Skolem functions such that if ψ results from φ' by replacing variables which appear existentially quantified in Q by terms denoting these Skolem functions then ψ is true in ω for sufficiently large values assigned to the variables which appear universally quantified in Q.*

PROOF OF THEOREM 2. Assume φ is an AE special sentence such that ZFO $\vdash [\Delta \models \varphi]$, but that no Horn reduct of φ is true in ω. Then by Lemma 2 there is a $p \geqslant 1$ such that ZFO $\vdash [\Delta \models \theta_p]$. Since θ_p is a positive sentence we may apply Proposition C and find almost combinatorial $f_j, j < 3$, an $i < p$, and $k \in \omega$ such that

$$2f_0(x_0, \ldots, x_{p-1}) + f_1(x_0, \ldots, x_{p-1}) = \sum_{j<p} x_j, \tag{17}$$

$$x_i + f_2(x_0, \ldots, x_{p-1}) = 3f_0(x_0, \ldots, x_{p-1}), \tag{18}$$

whenever $k \leqslant x_j < \omega$ for each $j < p$. Without loss of generality we may assume that k is sufficiently large so that each

$$f'_j = (\lambda x_0, \ldots, x_{p-1}) f_j(x_0 + k, \ldots, x_{p-1} + k)$$

is a combinatorial function. For $x \in \omega$ we define $f_j''(x) = f_j'(y_0, \ldots, y_{p-1})$ where $y_i = x$ and $y_q = 0$ for $i \neq q < p$. Substituting in (17), (18) we obtain

$$2f_0''(x) + f_1''(x) = x + kp, \tag{19}$$

$$x + k + f_2''(x) = 3f_0''(x), \tag{20}$$

for each $x \in \omega$. Expand the f_j'' in terms of their Stirling coefficients as $f_j''(x) = \sum_q c_q^j \binom{x}{q}$ and then use (19) and the uniqueness of these expansions to conclude that $2c_q^0 + c_q^1 = 0$ for $q > 1$ and $2c_1^0 + c_1^1 = 1$. Since the f_j'' are are combinatorial, each $c_q^j \in \omega$, implying that $c_q^0 = 0$ for $q > 0$. Thus f_0'' reduces to a constant contradicting (20). Q.E.D.

The contradiction used to prove Theorem 1 consisted in producing a y which was both finite and infinite, while for Theorem 2 we produced an f_0'' which was both bounded and unbounded. We have known for some time (cf. [3]) that unbounded combinatorial functions can be naturally associated with infinite Dedekind cardinals. What we do not know is how indecomposability enters into the picture.

We start our proof of Theorems 3 and 4 with

LEMMA 4. *Assume $x \in \Delta^*$ and $0 < n < \omega$. In ZF we can prove that if $nx \in \Delta$ then $x \in \Delta$.*

PROOF. (7) is a special case of the more general

$$nx \leqslant ny \to x \leqslant y \quad \text{for } 0 < n < \omega \tag{21}$$

which Δ also satisfies (cf. [8]). Thus suppose that $x = z - y$ where $y, z \in \Delta$ and $nx = nz - ny \in \Delta$. Then $ny \leqslant nz$ and hence by (21), $y \leqslant z$, i.e., $z - y \in \Delta$. Q.E.D.

PROOF OF THEOREMS 3 AND 4. Suppose $x_0, \ldots, x_{m-1} \in \Delta$ are not strongly linearly independent (mod n) in Δ^*. Then there exist $a_0, \ldots, a_{m-1} \in \omega$, not all congruent to 0 (mod n) and a $y \in \Delta^*$ such that

$$\sum_{i<m} a_i x_i = ny. \tag{22}$$

By the remainder theorem in ω we can write $a_i = q_i n + a_i'$ where $0 \leqslant a_i' < n$. Substituting in (22) gives $\sum_{i<m} a_i' x_i = n(y - \sum_{i<m} q_i x_i)$. Thus in (22) we may assume that $0 \leqslant a_i < n$ for $i < m$. Now the left-hand side of (22) is in Δ and hence $y \in \Delta$ by Lemma 4. We have shown that (22) holds in Δ where $a_0, \ldots, a_{m-1} \in \omega$, each $a_i < n$, but not all are equal to 0.

Now let Q be either ZF or ZFO. In order to show that $Q + \{\Delta^* \vDash \psi_{mn} | m > 0, n > 1\}$ is consistent we must show that no finite disjunction of the $[\Delta^* \vDash \sim \psi_{mn}]$ is a theorem of Q. Arguing for a contradiction, suppose that some such sentence is a theorem of Q. Since $\Delta^* \vDash \sim \psi_{mn}$ asserts that no sequence of m elements in Δ^* is strongly linearly independent (mod n), the analysis of the preceding paragraph implies that $Q \vdash [\Delta \vDash \varphi]$ where φ is a finite disjunction of sentences each having the form

$$(\forall u_0, \ldots, u_{m-1})(\exists v) \bigvee \sum_{i<m} a_i u_i = nv, \tag{23}$$

the disjunction in (23) being over all sequences $a_0, \ldots, a_{m-1} \in \omega$, each $a_i < n$, but not all 0. Now φ is a sentence to which Theorems 1 and 2 will apply. Thus if $Q \vdash [\Delta \models \varphi]$ there is a Horn reduct $(\forall u_0, \ldots, u_{m-1})$ $(\exists v) \sum_{i<m} a_i u_i = nv$ which is true in ω. Substituting the value 0 for each u_j except for one for which $a = a_i > 0$ we obtain a congruent to $0 \pmod n$ which is clearly false since $0 < a < n$. Q.E.D.

Note that this argument together with an isolic analogue of Theorems 1 and 2 (cf. [4]) gives a quick proof that each ψ_{mn} is true in the isolic integers.

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