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DEDEKIND SUMS AND NONCONGRUENCE SUBGROUPS OF THE HECKE GROUPS $G(\sqrt{2})$ AND $G(\sqrt{3})$

L. ALAYNE PARSON

ABSTRACT. An example is given of a character χ_r on a subgroup of $G(\sqrt{2})$ or $G(\sqrt{3})$ such that the kernel of χ_r is of finite index in $G(\sqrt{2})$ or $G(\sqrt{3})$ but is not a congruence subgroup.

In [7] K. Wohlfahrt exhibited a class of subgroups of the modular group which were not congruence subgroups although they were of finite index. These subgroups were the kernels of certain characters on $\Gamma_0(n)$. In this note we generalize K. Wohlfahrt's construction to the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$ in order to produce noncongruence subgroups of finite index in addition to those in [5] and to give examples of characters which are not congruence characters. Since $G(\sqrt{2})$ and $G(\sqrt{3})$ are the only Hecke groups commensurable with the modular group [3], K. Wohlfahrt's method cannot be extended to other Hecke groups.

For notational convenience let m stand for 2 or 3. Then $G(\sqrt{m})$ is the group of 2×2 matrices generated by

$$S = \begin{pmatrix} 1 & \sqrt{m} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is well known [1], [8] that $G(\sqrt{m})$ consists of the entirety of all matrices of the following two types:

$$\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}, \quad a, b, c, d \in Z, ad - mbc = 1,$$

and

$$\begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}, \quad a, b, c, d \in Z, mad - bc = 1.$$

For n a positive integer, the principal congruence subgroup of level n is defined by

$$\Gamma(n) = \{M \in G(\sqrt{m}) \colon M \equiv \pm I \pmod{n}\}$$

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where the congruence is elementwise and takes place in $Z[\sqrt{m}]$. A subgroup Γ of $G(\sqrt{m})$ is called a congruence subgroup of level n if $\Gamma(n) \subset \Gamma$ and n is minimal with respect to this property. We are particularly interested in the congruence subgroup $\Gamma_0(n)$ of level n defined by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\sqrt{m}) \colon \gamma \equiv 0 \pmod{n} \right\}$$

where the congruence again takes place in $Z[\sqrt{m}]$.

If Γ is a subgroup of finite index with $-I \in \Gamma$, a mapping v from Γ into the complex numbers of absolute value one satisfying $v(-I) = \exp(-\pi i r)$ and the "consistency condition" (1) is called a multiplier system for Γ of degree $-r$, r a real number.

$$\text{(1)} \qquad v(M_1 M_2)(\gamma_3 z + \delta_3)^r = v(M_1)v(M_2)(\gamma_1 M_2 z + \delta_1)^r(\gamma_2 z + \delta_2)^r$$

for z in the upper half plane, $M_1 = \begin{pmatrix} * & * \\ \gamma_1 & \delta_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} * & * \\ \gamma_2 & \delta_2 \end{pmatrix}$ in Γ and $M_1 M_2 = \begin{pmatrix} * & * \\ \gamma_3 & \delta_3 \end{pmatrix}$. Here $M_2 z = (a_2 z + b_2)/(c_2 z + d_2)$. To fix the branch of $(cz + d)^r$ for r nonintegral, for any complex number τ and real s we set $\tau^s = |\tau|^s \cdot \exp(is \arg \tau)$ with $-\pi \leqslant \arg \tau < \pi$. When r is an integer, (1) reduces to $v(M_1 M_2) = v(M_1)v(M_2)$; and v is a character on Γ. In [6] J. R. Smart has shown that if v is a multiplier system for the full group $G(\sqrt{m})$ of degree $-r$, then

$$\text{(2)} \qquad v(M) = v_{s,t}(M)v(M, \sqrt{m}, -r), \quad M \in G(\sqrt{m})$$

where $v_{s,t}$ is one of the $4m$ characters on $G(\sqrt{m})$ determined by M. I. Knopp in [2] and $v(M, \sqrt{m}, -r)$ is the multiplier system for $\eta(z, \sqrt{m})^r$. An explicit expression involving Dedekind sums is given for $v(M, \sqrt{m}, -r)$ in [6].

Now set $R = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$. It is easily verified that $R\Gamma_0(n)R^{-1} \subset G(\sqrt{m})$. For v a multiplier system of degree $-r$ on $G(\sqrt{m})$, we define v_R on $\Gamma_0(n)$ by $v_R(M) = v(RMR^{-1})$. It is then easy to check that v_R is a multiplier system of degree $-r$ on $\Gamma_0(n)$. Now set $\chi_r = v_R v^{-1}$. χ_r is a multiplier system of degree 0 and hence a character on $\Gamma_0(n)$. From (2) we have the following expression for χ_r. If $M = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$,

$$\text{(3)} \quad \chi_r(M) = \begin{cases} \exp[\pi i t b d(n-1)/m + \pi i r(m+1)(n-1)b/12d] & \text{if } c = 0, \\ \exp[\pi i t(n-1)(bd + ac/n)/m] & \\ \quad \cdot \exp\left[\dfrac{\pi i r}{12}(n-1)(m+1)(a+d)/mc \right. & \\ \qquad \left. + 12(\operatorname{sign} c)(s(a,c) + s(a,mc) - s(a,c/n) - s(a,mc/n))\right] & \text{if } c \neq 0. \end{cases}$$

If $M = \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}$,

$$\chi_r(M) = \exp[\pi it(1-n)(bd+ac/n)/m] \cdot \exp\left[\frac{\pi ir}{12}\{(n-1)(m+1)(a+d)/c + 12(\operatorname{sign} c)(s(ma,c)+s(a,c)-s(ma,c/n)-s(a,c/n))\}\right].$$

Here t is an integer, $0 \leqslant t \leqslant m-1$, and $s(h,k)$ is Dedekind's sum. We note that if r is rational, then $\chi_r(M)$ is always a root of unit. In this case, the kernel of χ_r is a finite index in $\Gamma_0(n)$ and therefore in $G(\sqrt{m})$.

THEOREM. *If $n > 1$ and if r is rational, $r = h/k$, $(h,k) = 1$, with k chosen so that $6n^2(n-1) \not\equiv 0 \pmod k$, then χ_r is not a congruence character on $\Gamma_0(n)$, that is, the kernel K_r of χ_r is not a congruence subgroup.*

PROOF. By way of contradiction, we assume that χ_r is a congruence character. In [4] it is shown that any congruence character on a congruence subgroup of level n is identically one on $\Gamma(48n^2)$ when $m=2$ and $\Gamma(36n^2)$ when $m=3$. In particular, we must have $\chi_r(M)=1$ when $M = \left(\begin{smallmatrix}1 & 48n^2\sqrt{2}\\ 0 & 1\end{smallmatrix}\right)$ or $M = \left(\begin{smallmatrix}1 & 36n^2\sqrt{3}\\ 0 & 1\end{smallmatrix}\right)$. From (3) we have that

$$\chi_r(M) = \exp[12\pi ih(n-1)n^2/k].$$

However, by the choice of k, $\chi_r(M) \neq 1$; and we have the necessary contradiction.

REMARKS. For fixed n, there exist infinitely many primes p such that $r = 8/p$ for $m=2$ and $r = 6/p$ for $m=3$ satisfy the conditions of the Theorem. For these values of r, if we take $t=0$, then the K_r are distinct since the smallest power of S in K_r with strictly positive exponent is S^p. This gives infinitely many subgroups of finite index which are not congruence subgroups.

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210