THE LEVI FORM AND LOCAL COMPLEX FOLIATIONS

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ABSTRACT. A short coordinate-free proof is given for some known results on the existence of local complex-analytic foliations of a real submanifold M of C^n . The proof uses an explicit formulation of the equivalence between two definitions of the E. E. Levi form of M.

A local definition will suffice for the submanifold; $M = \{z \in U: \rho(z) = 0\}$, where $\rho = (\rho_1, \ldots, \rho_m): U \to \mathbb{R}^m$ is part of a smooth real coordinate patch for U open in \mathbb{C}^n . \mathcal{C} is the ring of smooth complex-valued functions on U and $T\mathbb{C}^n$ the \mathcal{C} -module of smooth vector fields on U. Geometric objects on U are regarded as residue classes X' = X + O(M) of objects X on U, where O(M) is the subspace of objects vanishing on M. Thus the ring \mathcal{C}' of smooth functions on M is $\mathcal{C}' = \mathcal{C}/O(M)$ and the \mathcal{C}' -module of smooth vector fields on U is U in U

The E. E. Levi form of M is a bilinear map of $H' \times H'$ which can be conceived in (at least) two ways:

(1) as $L\rho: H' \times H' \to \mathcal{C}'^m$ defined by

$$L\rho(X',Y') = \partial \bar{\partial} \rho(X,Y)' = (\partial \bar{\partial} \rho_1(X,Y)',\ldots,\partial \bar{\partial} \rho_m(X,Y)'),$$

or

(2) as $L: H' \times H' \to T'/H'$ defined by L(X', Y') = [X', Y'] + H', where [X', Y'] = [X, Y]' for $X', Y' \in T'$ [5].

That (1) and (2) are essentially equivalent has been known for some time. Recently, an explicit coordinate-free expression of their relationship was suggested in [1, proof of Proposition 3.2.1]. Due to its usefulness, this fundamental fact deserves more emphasis:

(3) The differential $\bar{\partial}\rho$ induces a \mathcal{C}' -monomorphism $\alpha\colon T'/H'\to \mathcal{C}'^m$ such that $L\rho(X',Y')=\alpha L(X',Y'),X',Y'\in H'$.

This is proved by applying the standard identity $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ to $\omega = \bar{\partial}\rho$ and $X, Y \in H$. It is easy to see that the first two terms on the right vanish on M and conclude that

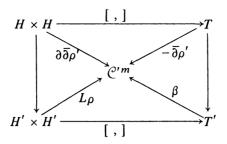
(4) $\partial \bar{\partial} \rho(X, Y)' = -\bar{\partial} \rho([X, Y])', X, Y \in H$ (recall that $d\bar{\partial} \rho_j = \partial \bar{\partial} \rho_j$). Relation (4) says that the top triangle of the diagram below commutes.

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The vertical maps are natural projections and $\beta(X') = -\bar{\partial}\rho(X)'$. Since the right and left triangles and the outer square commute, a simple chase of this diagram shows

(5) $L\rho(X', Y') = \beta([X', Y']), X', Y' \in H'.$

Now $\ker(\beta|T') = H'$ because $d\rho = \partial\rho + \bar{\partial}\rho$. Therefore there exists a unique monomorphism α : $T'/H' \to \mathcal{C}'^m$ such that $\alpha \circ \pi = \beta$, where π : $T' \to T'/H'$ is the quotient map. This and (5) show that α satisfies (3).

Relation (3) permits an easy proof of the existence of local complex foliations of M. Consider the null space $N' = \{X' \in H' : L\rho(X', Y') = 0 \forall Y' \in H'\}$, which is equivalent to the Levi null space defined pointwise in [2, Definition 2.7] under the constant rank assumptions made there. It requires a lot of calculation in [2] to prove (under these assumptions), a result [2, Theorem 6.1C] equivalent to the integrability condition $[N', N'] \subset N'$, yielding a foliation by complex submanifolds tangent to N'. This integrability condition can be obtained directly by using (3) to rewrite N' as $N' = \{X' \in H' : [X', Y'] \in H'$ for every $Y' \in H'\}$ and then simply inspecting the Jacobi identity [[X', Y'], Z'] + [[Z', X'], Y'] + [[Y', Z'], X'] = 0. It is clear that the Levi flat case N = H is described by the equivalent conditions $L\rho = 0$ and $[H', H'] \subset H'$, due to Sommer [4] when M is a hypersurface. His proof and a later one in [3] require considerable calculation in coordinates.

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