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THE LEVI FORM AND LOCAL COMPLEX FOLIATIONS

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ABSTRACT. A short coordinate-free proof is given for some known results on the existence of local complex-analytic foliations of a real submanifold M of $\mathbf{C}^n$. The proof uses an explicit formulation of the equivalence between two definitions of the E. E. Levi form of M.

A local definition will suffice for the submanifold; $M = \{z \in U\colon \rho(z) = 0\}$, where $\rho = (\rho_1, \ldots, \rho_m)\colon U \to \mathbf{R}^m$ is part of a smooth real coordinate patch for U open in $\mathbf{C}^n$. $\mathcal{C}$ is the ring of smooth complex-valued functions on U and $T\mathbf{C}^n$ the $\mathcal{C}$-module of smooth vector fields on U. Geometric objects on M are regarded as residue classes $X' = X + O(M)$ of objects X on U, where $O(M)$ is the subspace of objects vanishing on M. Thus the ring $\mathcal{C}'$ of smooth functions on M is $\mathcal{C}' = \mathcal{C}/O(M)$ and the $\mathcal{C}'$-module of smooth vector fields on M is $T' = T/O(M)$, where $T = \{X \in T\mathbf{C}^n\colon X\rho_j = d\rho_j(X) \in O(M), \text{ all } j\}$. The submodule of complex tangent vector fields is $H' = H/O(M)$, where $H = \{X \in T\mathbf{C}^n\colon \partial\rho_j(X) \text{ and } \bar\partial\rho_j(X) \in O(M), \text{ all } j\}$.

The E. E. Levi form of M is a bilinear map of $H' \times H'$ which can be conceived in (at least) two ways:

(1) as $L\rho\colon H' \times H' \to \mathcal{C}'^m$ defined by

$$L\rho(X', Y') = \partial\bar\partial\rho(X, Y)' = \big(\partial\bar\partial\rho_1(X, Y)', \ldots, \partial\bar\partial\rho_m(X, Y)'\big),$$

or

(2) as $L\colon H' \times H' \to T'/H'$ defined by $L(X', Y') = [X', Y'] + H'$, where $[X', Y'] = [X, Y]'$ for $X', Y' \in T'$ [5].

That (1) and (2) are essentially equivalent has been known for some time. Recently, an explicit coordinate-free expression of their relationship was suggested in [1, proof of Proposition 3.2.1]. Due to its usefulness, this fundamental fact deserves more emphasis:

(3) *The differential $\bar\partial\rho$ induces a $\mathcal{C}'$-monomorphism $\alpha\colon T'/H' \to \mathcal{C}'^m$ such that $L\rho(X', Y') = \alpha L(X', Y')$, $X', Y' \in H'$.*

This is proved by applying the standard identity $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ to $\omega = \bar\partial\rho$ and $X, Y \in H$. It is easy to see that the first two terms on the right vanish on M and conclude that

(4) $\partial\bar\partial\rho(X, Y)' = -\bar\partial\rho([X, Y])'$, $X, Y \in H$

(recall that $d\bar\partial\rho_j = \partial\bar\partial\rho_j$). Relation (4) says that the top triangle of the diagram below commutes.

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$$
\begin{tikzcd}
H \times H \arrow[rr, "{[\ ,\ ]}"] \arrow[dd] \arrow[dr, "\partial\bar{\partial}\rho'"'] & & T \arrow[dd] \arrow[dl, "-\bar{\partial}\rho'"] \\
& \mathcal{C}'^{m} & \\
H' \times H' \arrow[rr, "{[\ ,\ ]}"'] \arrow[ur, "L\rho"] & & T' \arrow[ul, "\beta"']
\end{tikzcd}
$$

The vertical maps are natural projections and $\beta(X') = -\bar{\partial}\rho(X)'$. Since the right and left triangles and the outer square commute, a simple chase of this diagram shows

(5) $L\rho(X', Y') = \beta([X', Y'])$, $X', Y' \in H'$.

Now $\ker(\beta | T') = H'$ because $d\rho = \partial\rho + \bar{\partial}\rho$. Therefore there exists a unique monomorphism α: $T'/H' \to \mathcal{C}'^{m}$ such that $\alpha \circ \pi = \beta$, where π: $T' \to T'/H'$ is the quotient map. This and (5) show that α satisfies (3).

Relation (3) permits an easy proof of the existence of local complex foliations of M. Consider the null space $N' = \{X' \in H'\colon L\rho(X', Y') = 0\ \forall Y' \in H'\}$, which is equivalent to the Levi null space defined pointwise in [2, Definition 2.7] under the constant rank assumptions made there. It requires a lot of calculation in [2] to prove (under these assumptions), a result [2, Theorem 6.1C] equivalent to the integrability condition $[N', N'] \subset N'$, yielding a foliation by complex submanifolds tangent to N'. This integrability condition can be obtained directly by using (3) to rewrite N' as $N' = \{X' \in H'\colon [X', Y'] \in H'$ for every $Y' \in H'\}$ and then simply inspecting the Jacobi identity $[[X', Y'], Z'] + [[Z', X'], Y'] + [[Y', Z'], X'] = 0$. It is clear that the *Levi flat* case $N = H$ is described by the equivalent conditions $L\rho = 0$ and $[H', H'] \subset H'$, due to Sommer [4] when M is a hypersurface. His proof and a later one in [3] require considerable calculation in coordinates.

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