PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 2, June 1976

COUNTABLE SPACES HAVING EXACTLY ONE NONISOLATED POINT. I

FRANK SIWIEC

ABSTRACT. Some countable, T_1-, k-spaces having exactly one nonisolated point are characterised by means of intrinsic properties and mapping conditions.

1. Four easy examples.

EXAMPLE A. Let $X = \{0, 1, 1/2, 1/3, \ldots\}$ with the usual relative topology. This is the unique example (up to homeomorphism) of a countable, compact, T_1-space with exactly one nonisolated point.

EXAMPLE B. Let $X = \{0, 1, 1/2, 2, 1/3, 3, 1/4, 4, \ldots\}$ with the usual relative topology. This is the unique example of a countable, locally compact, noncompact, T_1-space with exactly one nonisolated point.

EXAMPLE C. Let X be the subspace $\{(0,0)\} \cup \{(1/m, 1/n) | m, n \in N\}$ of the plane, where N is the set of natural numbers. Equivalently, X is $\{0\} \cup \{1/m + 1/n | m, n \in N \text{ and } n \geqslant m\}$ as a subspace of the line. Equivalently, X is the space of rational numbers with the topology enlarged by having each nonzero point also an open set. (Still another description may be found in [1].) This space X is completely metrizable and not locally compact. It is an easy exercise to show that this is the unique example of a countable, first countable, T_1-space having exactly one nonisolated point, which is not locally compact.

The above Example C is also not hemicompact. (A space X is said to be *hemicompact* if X has a countable cover $\mathcal{C}$ of compact subspaces such that if K is a compact subset of X then there exists a $C \in \mathcal{C}$ for which $K \subset C$. Every Lindelöf locally compact space is hemicompact.) A closely related concept is that of a k_ω-space. (A space X is a *k_ω-space* if X has a countable cover $\mathcal{C}$ of compact subspaces such that a set $A \subset X$ is closed whenever $A \cap C$ is closed in C for every $C \in \mathcal{C}$.) Since a Hausdorff space is a k_ω-space if and only if it is a hemicompact k-space, Example C is also not a k_ω-space. Our first three examples were metrizable; the following is not.

EXAMPLE D. Consider the discrete union of countably many copies of Example A, and let X be the space obtained by identifying the nonisolated points to one point. This example, commonly called the sequential fan, is the unique example of a countable, hemicompact, T_1-, k-space having exactly one

Received by the editors April 9, 1974 and, in revised form, February 15, 1975 and December 1, 1975.

AMS (MOS) subject classifications (1970). Primary 54F65.

Key words and phrases. Countable space, space having exactly one nonisolated point, σMK space.

nonisolated point, which is not locally compact, as may be seen from Theorem 1.2.

Professor Rajagopalan has asked that it be pointed out that Professor V. Kannan should be given credit for showing [2] that Examples A, C, and D are nonhomeomorphic.

PROPOSITION 1.1. *Let X be a countable, regular, T_1, Fréchet space. Then the following are equivalent.*

(a) *X is a hemicompact space.*

(b) *X is a k_ω-space.*

(c) *X has a hereditarily closure-preserving cover of compact sets.*

(d) *X is a closed image of a locally compact, countable metric space.*

(e) *X is a closed image of a locally compact metric space.*

(f) *X is a quotient image of a hemicompact k-space.*

PROOF. Some of the conditions are equivalent under more general hypotheses. Let (a_k) be the condition that X is a hemicompact k-space. That conditions (a_k) and (b) are equivalent was mentioned above, and it is clear that (a_k) implies (f). Morita [3] has shown that the property of a space being a k_ω-space is preserved to the image by quotient maps. (All maps are continuous, onto functions.) Thus, (f) implies (b). As a result, conditions (a_k), (b), and (f) are equivalent for any (Hausdorff) space.

Let (a_m) be condition (a) together with the condition that every compact subspace of X is metrizable. Define (c_m) likewise. It is immediate from the proof of Theorem 7 of Telgarsky [7] that conditions (c_m) and (e) are equivalent for any (Hausdorff) space.

If X is a countable space, then condition (e) implies condition (f). For the proof, let f be a closed map of a locally compact metric space M onto X, and for each point x in X, choose a point p_x in $f^{-1}(x)$. Let M' be the closure of the set $\{p_x | x \in X\}$ in M, and let $f' = f|M'$. Then f' is a closed map of the locally compact, separable, metrizable space M' onto X. Thus, (e) implies (f).

It is shown in [6] that for a Hausdorff Fréchet space, (a_m) implies (e). It is easy to see that for a countable space, (c) implies (d). Of course (d) implies (e).

Under the stated hypotheses of this proposition, conditions (a) through (f) are now seen to be equivalent.

THEOREM 1.2. *Let X be a countable, T_1-space having exactly one nonisolated point. Then X satisfies one of the conditions of Proposition* 1.1 *if and only if X is one of the Examples* A, *or* B, *or* D.

PROOF. We need only prove that condition (a) of Proposition 1.1 implies that the space X is one of A, or B, or D. If X is compact, then X is Example A. Now suppose that X is not compact, but X is a hemicompact k-space. Then X is the union of an increasing sequence $\{K_i\}$ of compact sets such that if K is compact then there is an i for which $K \subset K_i$. Let x_0 denote the nonisolated point of X. We may assume that each K_i contains x_0. There are infinitely many distinct K_i, for otherwise X is compact. Let $K_1' = K_1$, and $K_i' = (K_i - K_{i-1}) \cup \{x_0\}$ for $i = 2, 3, \ldots$. Let $X_1 = \cup\{K_i' | K_i'$ is infinite$\}$, and $X_2 = \cup\{K_i' | K_i'$ is finite$\}$. Then X_2 is either empty or a discrete subspace of X. For suppose that

X_2 is not empty and contains an infinite compact set K. Then K meets infinitely many K_i'. But since K is compact, there is a j for which $K \subset K_j$, and so $K \subset \bigcup\{K_i' | i \leqslant j\}$. Since this is impossible, every compact subspace of X_2 is finite. But a nonempty, T_1-, k-space in which every compact set is finite is a discrete space. Thus X_2 is either empty of discrete. Since each K_i' forming X_1 is a convergent sequence, X_1 must be either Example A or Example D. Thus X is Example A, B, or D.

2. Two more examples.

EXAMPLE E. Let X be the discrete union of Examples C and D with the nonisolated points identified to one point. This example is a nonmetrizable, nonhemicompact space, which is a closed image of a countable metric space.

EXAMPLE F. Let X be the discrete union of countably many copies of Example C with the nonisolated points identified to one point. Equivalently, X is the space of rational numbers with the integers identified to one point and with the topology enlarged by having each nonzero point also an open set. This example is also a nonmetrizable, nonhemicompact space, which is a closed image of a countable metric space.

PROPOSITION 2.1. *Examples* E *and* F *are distinct. In fact, a quotient image of Example* E *cannot contain a subspace homeomorphic to Example* F.

PROOF. Let $X = (X_1 + X_2)/x_0$ be Example E, where X_1 is Example C (with its metric denoted by d), X_2 is Example D, the plus sign denotes the discrete union, and x_0 denotes the nonisolated point of X_1 and of X_2. Let f be a quotient map of X onto a space Y. Suppose that Y is not discrete, and let y_0 denote the nonisolated point of Y. Now suppose that Y contains a sequence $\{Y_n\}$ of subspaces such that each Y_n is homeomorphic to Example C and $Y_m \cap Y_n = \{y_0\}$ for $m \neq n$. Notice that for each n, $[f^{-1}(Y_n)] \cap X_1$ is homeomorphic to Example C. So for each n, there exists a point x_n in $[f^{-1}(Y_n)] \cap X_1$ for which $d(x_n, x_0) < 1/n$ in X_1. Then $x_n \to x_0$ in X_1. As a result $f(x_n) \to y_0$ in Y. But $f(x_n) \in Y_n$ for all n. Thus $\bigcup_n Y_n$ cannot be homeomorphic to Example F.

3. A new concept.

Before stating our main result, we briefly discuss a new topological concept. Assume all spaces are regular and T_1 in this paragraph. We then define a space X to be σMK provided that X has a countable cover $\mathcal{C}$ of closed metrizable subspaces such that if K is a compact subset of X, there is a $C \in \mathcal{C}$ for which $K \subset C$. This property of a space is hereditary, is preserved to the image by perfect maps, is preserved by finite closed unions, and is preserved by finite products, but not preserved by countable products [4]. A hemicompact space, in which every compact subspace is metrizable, is σMK. A σMK is an $\aleph$-space in the sense of O'Meara [5], and a first countable σMK space is metrizable.

I am indebted to Professor V. Kannan for finding an error in the following theorem as stated in a previous draft of this paper.

THEOREM 3.1. *Let* X *be a countable* T_1*-space having exactly one nonisolated point. Then the following are equivalent.*

(a) X *is one of the six Examples* A, B, C, D, E *or* F.

(b) X *is* σMK *and a* k*-space.*

(c) *X has a countable closed cover $\mathcal{C}$ of metrizable subspaces such that a set $A \subset X$ is closed whenever $A \cap C$ is closed in C for every $C \in \mathcal{C}$.*

(d) *$X = M/A$, where M is a countable metric space, A is a locally compact subspace, $M - A$ is a dense subspace, and each point of $M - A$ is an open subset of M.*

(e) *X is a closed image of one of the six Examples* A, B, C, D, E *or* F.

PROOF. It is clear that (a) implies (b) and it is not hard to show that conditions (b) and (c) are equivalent.

(b) implies (a). Suppose that X is not homeomorphic to any of the Examples A, B, C, D, or E. Let $X = \cup\{C_n | n \in N\}$ as given by the σMK condition. We may assume that the nonisolated point x_0 is in C_1, that C_1 is nondiscrete, and that $C_n \subset C_{n+1}$ for all n. Then each C_n is homeomorphic to Example A, B, or C. If no C_n is homeomorphic to Example C, then X is easily seen to be one of the Examples A, B, or D, by Theorem 1.2. Thus, we may assume that each C_n is homeomorphic to Example C.

Now, let $X'_1 = C_1$. If $(C_2 - X'_1) \cup \{x_0\}$ is homeomorphic to Example C, let $X'_2 = (C_2 - X'_1) \cup \{x_0\}$ and let $X_2 = X'_1$. If $(C_2 - X'_1) \cup \{x_0\}$ is not homeomorphic to Example C, let $X'_2 = C_2 \cup X'_1$ and let X_1 be the empty set. In general, if

$$D_n = (C_n - X_1 - X_2 - \cdots - X_{n-2} - X'_{n-1}) \cup \{x_0\}$$

is homeomorphic to Example C, let X'_n be D_n, and let $X_{n-1} = X'_{n-1}$. If D_n is not homeomorphic to Example C, let $X'_n = (C_n - \cup_{m \leqslant n-2} X_m) \cup \{x_0\}$ and let X_{n-1} be the empty set. There are infinitely many nonempty X_n, for otherwise X is homeomorphic to Example C or E. It is clear that if $m \neq n$, then $X_m \cap X_n$ either equals $\{x_0\}$ or is empty. Let $\{X_{n_i} | i \in N\}$ be the collection of all those X_{n_i} which are nonempty and let X^i denote X_{n_i} for all i. Then each X^i is homeomorphic to Example C, and $X = \cup_i X^i$.

If S is a sequence in X converging to x_0, and $x_0 \in S$, then there exists an index i_0 for which $S \subset X^1 \cup \cdots \cup X^{i_0}$. To see this note that there exists an index n for which $S \subset C_n$. Since $C_n \subset X_1 \cup \cdots \cup X_{n-1} \cup X'_n$ and there exists an $i_0 \geqslant n$ for which $X'_n \subset X_{i_0}$, we have that

$$S \subset X_1 \cup \cdots \cup X_{i_0} \subset X^1 \cup \cdots \cup X^{i_0}.$$

In order to show that X is homeomorphic to Example F, we need to show that the neighborhoods of x_0 in X are identical to those of Example F. Let G be an open neighborhood of x_0 in the topology of X. Then $G \cap X^i$ is open in X^i (= Example C) for all i. Thus G is open in the topology of Example F. Conversely, let G be an open neighborhood of x_0 in the topology of Example F. Then $G \cap X^i$ is open in X^i (= Example C) for all i. Suppose that G is not open in the topology of X. Then $X - G$ is not closed so x_0 is an accumulation point of $X - G$. Then there exists a sequence S in $(X - G) \cup \{x_0\}$, with $x_0 \in S$, and with S converging to x_0. By the above, there exists an index i_0 for which $S \subset X^1 \cup \cdots \cup X^{i_0}$. Since S converges to x_0, and $G \cap X^i$ is open in X^i for all i, S meets G in infinitely many points. This is impossible, since $S \subset (X - G) \cup \{x_0\}$. The proof of (b) implies (a) is complete.

Clearly (a) implies (d). For the converse, if A is finite, then clearly X is metrizable. Now assume A is infinite and write $A = \{p_n | n \in N\}$. Let G'_n be open in A such that $p_n \in G'_n$ and G'_n has compact closure. Let G_n be open in M such that $G'_n = G_n \cap A$. We may assume that each $G_n \subset G_{n+1}$, that each G_n is in fact both open and closed, and that $\bigcup\{G_n | n \in N\} = M$. Then each $f(G_n)$ is a closed metrizable subspace of X and $X = \bigcup\{f(G_n) | n \in N\}$. Thus X is σMK and we have that (d) implies (b).

It is clear that (a) implies (e). We show that (e) implies (b). Let Z be one of the stated six examples and let f be a closed map of Z onto X. Let z_0 denote the nonisolated point of Z. We may assume that f is one-to-one on $f^{-1}(X - \{x_0\})$. Let $F = f^{-1}(x_0)$. Since Z is σMK and Fréchet, by [6], there exists a metric space M and a map g of M onto Z such that g is closed, g is one-to-one on $g^{-1}(Z - \{z_0\})$, and $\operatorname{Bdy} g^{-1}(z_0)$ is hemicompact in M. Then $g \circ f$ is a closed map of M onto X. Also $g \circ f$ is one-to-one on $(g \circ f)^{-1} \cdot (X - \{x_0\})$. And

$$(g \circ f)^{-1}(x_0) = g^{-1}(F) = g^{-1}(z_0) \cup g^{-1}(F - \{z_0\}).$$

But $F - \{z_0\}$ is open in Z, so that $\operatorname{Bdy}(g \circ f)^{-1}(x_0) = \operatorname{Bdy} g^{-1}(z_0)$ which is hemicompact. Again by the main theorem of [6], X is σMK and Fréchet. The proof is complete.

4. **Another example.** There are countable, T_1-, k-spaces having exactly one nonisolated point, which are not σMK. An easy example of such a space may be obtained by identifying the irrational numbers of the real line to one point and then enlarging the topology by having each rational point also an open set. If this space is covered by a countable number of metrizable subspaces, then one may construct a convergent sequence which is not contained in any of these metrizable subspaces. Thus the space is not σMK. This example is then a closed image of a separable metric space, but yet not one of the six examples discussed above.

5. **A remark on k-spaces and non-k-spaces.** Countable T_1-spaces having exactly one nonisolated point may be divided into three disjoint classes. Those which are k-spaces, those which contain no convergent sequence, and those which are neither of these. It is easy to see that every space of the latter class has its topology as an intersection of two topologies: a topology of a space of the first class (the topology generated by the sequentially-open sets) and a topology of a space of the second class (the topology obtained by taking all convergent sequences, with their limit deleted, as additional closed sets).

REFERENCES

1. James R. Boone, *A space in which the convergence criterion determines the metric*, 1970 (unpublished).

2. V. Kannan, *An extension that nowhere has the Fréchet property*, Michigan Math. J. **20** (1973), 225–234. MR **48** #3006.

3. Kiiti Morita, *On decomposition spaces of locally compact spaces*, Proc. Japan Acad. **32** (1956), 544–548. MR **19**, 49.

4. Keiô Nagami, *Dimension for σ-metric spaces*, J. Math. Soc. Japan **23** (1971), 123–129. MR **44** #4725.

5. Paul O'Meara, *On paracompactness in function spaces with the compact-open topology*, Proc. Amer. Math. Soc. **29** (1971), 183–189. MR **43** #2659.

6. F. Siwiec, *A note on identifications of metric spaces,* Proc. Amer. Math. Soc. 57 (1976), 340–344.

7. R. Telgársky, *Concerning two covering properties*, Colloq. Math. (to appear).

DEPARTMENT OF MATHEMATICS, JOHN JAY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10019