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THE SPACES lip α AND CERTAIN OTHER SPACES HAVE DUALS WITH CESÀRO BASES

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ABSTRACT. Banach sequence spaces whose duals are Banach sequence spaces with Toeplitz bases are characterized. For example, the duals of the lip α spaces, for $0 < \alpha < 1$, are shown to have Cesàro bases. Also reflexive spaces with a Toeplitz basis are characterized and an equivalent form of the well-known theorem of F. and M. Riesz on the absolute continuity of measures is given.

1. **Introduction.** Let $0 < \alpha < 1$, let $f_h(x) = f(x+h)$, and let $\|\cdot\|$ denote the sup norm of $L^\infty(0, 2\pi)$. Lip α is the space of all 2π-periodic complex valued functions f on $\mathbf{R}$ for which $\|f_h - f\| = O(|h|^\alpha)$ as $h \to 0$, and lip α is the space of all $f \in$ Lip α for which $\|f_h - f\| = o(|h|^\alpha)$ as $h \to 0$. Lip α and lip α are Banach spaces under the norm

$$\|f\|_\alpha = \max\left\{\|f\|, \sup_{h \neq 0} \frac{\|f_h - f\|}{|h|^\alpha}\right\}.$$

In [2], K. de Leeuw showed that the second dual (lip α)$''$ of lip α is Lip α. His proof also shows that the dual of lip α is separable. We prove general theorems on Banach sequence spaces which show, in particular, that (lip α)$'$ even has a Cesàro basis. In general, we characterize Banach sequence spaces whose duals are Banach sequence spaces with Toeplitz bases (§3). We apply these theorems to lip α in 4.1 and 4.2, to reflexive Banach spaces with Toeplitz bases in 4.3, and to the well-known F. and M. Riesz Theorem in 4.5.

This paper is a result of several discussions on the lip α spaces with Professor Günther Goes to whom I am also thankful for many suggestions.

2. **Definitions.** Let ω be the space of all complex valued sequences $x = (x_k)_{k=1}^\infty$ and let ϕ be the subspace of all sequences with a finite number of nonzero coordinates. Let $T = (t_{nk})$ be an infinite matrix with rows in ϕ and with columns converging to 1. For each $x \in \omega$, the nth T-section of x is $t^n x = (t_{nk} x_k)$. A BK-space is a subspace of ω which is a Banach space with continuous coordinates [10], [9, §§11.3, 12.4]. We suppose all BK-spaces considered contain ϕ.

Let E be a BK-space. E_{AD} is the closure of ϕ in E. E_{TB} is the space of all sequences x in ω (not necessarily in E) for which $\sup_n \|t^n x\|_E < \infty$ and E_{TK} is the space of all sequences x in E for which $\lim_n \|t^n x - x\|_E = 0$. E_{FTK}

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$= \{x \in \omega\colon \lim_n f(t^n x)$ exists for all $f \in E'\}$. Clearly $E_{TK} \subset E_{AD}$ and $E_{TK} \subset E_{FTK} \subset E_{TB}$. E is a TB-space (resp., a TK-space, an AD-space) if $E \subset E_{TB}$ (resp., $E = E_{TK}$, $E = E_{AD}$).

The condition TK says that $\{t^n\}_{n=1}^{\infty}$ is an approximate identity for E. This implies that the set $\{e^k\}_{k=1}^{\infty}$, where e^k is the sequence with 1 in the kth position and zero elsewhere, is a Toeplitz basis for E [5], [6, p. 45]. In particular, let $T = \sigma = (\sigma_{nk})$ be the matrix defined by $\sigma_{nk} = (1 - (k-1)/n)$ if $k \leqslant n$ and $\sigma_{nk} = 0$ if $k > n$. Then $\{e^k\}$ is a Cesàro basis (of order one) for E if and only if E is a σK-space. In this case, we say that E has Cesàro sectional convergence.

Let E be a BK-space. Each $f \in E'$ defines a sequence $y^f = (f(e^k))_{k=1}^{\infty}$. Thus E' is associated with the sequence space $E^{\phi} = \{y \in \omega\colon \exists f \in E' \ni y = y^f\}$. If the correspondence $f \to y^f$ is one-to-one, then we can identify E' with the sequence space E^{ϕ} and we write $E' \cong E^{\phi}$. Since E_{AD} is a BK-space and $(E_{AD})' \cong (E_{AD})^{\phi} = E^{\phi}$, E^{ϕ} is a BK-space under the dual norm of E_{AD} [1, Proposition 1]. If $E' \cong E^{\phi}$, then we identify E'' with the dual of the BK-space E^{ϕ}. Thus $E'' \cong (E^{\phi})^{\phi}$ denotes $(E^{\phi})' \cong (E^{\phi})^{\phi}$. We define E^{γ_T} (respectively, E^{β_T}) as the space of all y in ω for which $\sum_{k=1}^{\infty} t_{nk} x_k y_k$ is bounded (resp., converges as $n \to \infty$) for each x in E.

3. **Main results.**

3.1. PROPOSITION. *Let E be a BK-space. Then $E' \cong E^{\phi}$ if and only if E is an AD-space.*

PROOF. We have $E' \cong E^{\phi}$ if and only if every f in E' is determined by its values on e^k, $k = 1, 2, 3, \ldots$. Since ϕ is the span of $\{e^k\}_{k=1}^{\infty}$, this is equivalent to the property AD. □

3.2. COROLLARY. *Suppose E is a BK-space and an AD-space. Then $E'' \cong (E^{\phi})^{\phi}$ if and only if E^{ϕ} is an AD-space.*

3.3. THEOREM. *Let E be a BK-space. Consider the following statements.*
(a) *E is a TB-space.*
(b) *$E_{TB} = (E^{\gamma_T})^{\gamma_T}$.*
(c) *E^{ϕ} is a TB-space.*
(d) *E_{AD} is a TK-space.*
(e) *$E_{TB} = (E^{\phi})^{\phi}$.*
We have the following implications:

$$(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e).$$

If E is an AD-space, all statements are equivalent.

PROOF. (a)$\Leftrightarrow$(b) follows from Theorem 4 of [1]. A TB-space which is an AD-space is a TK-space [11, Satz 4]. Hence (a) $\Rightarrow$ (d). By [1, Theorem 1], $E_{TB} = (E^{\phi})^{\gamma_T}$ and by [1, Theorem 4], E^{ϕ} is a TB-space if and only if $(E^{\phi})^{\gamma_T} = (E^{\phi})^{\phi}$. Thus (c) $\Leftrightarrow$ (e). (c) $\Leftrightarrow$ (d) follows from $E^{\phi} = (E_{AD})^{\phi}$ and [1, Proposition 2]. If $E = E_{AD}$, then evidently (d) $\Rightarrow$ (a). □

3.4. REMARK. If E is not an AD-space the implication (c) $\Rightarrow$ (a) in Theorem 3.3 is false. For example, let E be the space spanned by the constant sequences and the space $l = \{x \in \omega: \sum |x_k| < \infty\}$. Then E is a BK-space under the norm given by $\|z\|_E = |c(z)| + \sum_k |z_k - c(z)|$ where $c(z) = \lim_k z_k$. Let $t_{nk} = 1$ if $k \leqslant n$ and $t_{nk} = 0$ if $k > n$. Then E is not a TB-space since $E_{AD} = E_{TK} = E_{TB} = l$. But $E^\phi = (E_{AD})^\phi = l^\phi = \{x \in \omega: \sup_k |x_k| < \infty\}$ is a TB-space.

3.5. COROLLARY. *Let E be a BK-space. Then E is a TK-space if and only if $E' \cong E^\phi$ and E^ϕ is a TB-space.*

3.6. THEOREM. *Suppose E is a BK-space and an AD-space. Then $E'' \cong (E^\phi)^\phi = E_{TB}$ if and only if E^ϕ is a TK-space.*

PROOF. E^ϕ is a TK-space if and only if it is an AD-space and a TB-space [11, Satz 4]. By 3.2, E^ϕ is an AD-space if and only if $E'' \cong (E^\phi)^\phi$. By 3.3, E^ϕ is a TB-space if and only if $(E^\phi)^\phi = E_{TB}$. □

4. **Applications.** Lip α and lip α can be identified with the BK-spaces $\widehat{\text{Lip}}\ \alpha$ and $\widehat{\text{lip}}\ \alpha$ consisting of the sequences $\hat{f}$ of Fourier coefficients of f in Lip α and lip α, respectively, when we define the norm by $\|\hat{f}\|_\alpha = \|f\|_\alpha$. These are spaces of sequences $x = (x_k)_{k=-\infty}^{k=+\infty}$ defined on the integers rather than the natural numbers. However, the results of §3 hold for these sequence spaces with minor changes in some definitions. In [7], it is shown that $(\widehat{\text{Lip}}\ \alpha)_{AD} = \widehat{\text{lip}}\ \alpha$. Thus, by 3.1, $(\text{lip}\,\alpha)'$ can be identified with the BK-space $(\widehat{\text{lip}}\ \alpha)^\phi$.

4.1. THEOREM. $(\text{lip}\,\alpha)'$ *is a Banach space with a Cesàro basis.*

PROOF. We show that $(\widehat{\text{lip}}\ \alpha)^\phi$ is a σK-space. Let $f \in \text{Lip}\,\alpha$. We define $\|\hat{f}\|_{\hat{L}^\infty} = \|f\|$, where $\|\cdot\|$ denotes the sup norm on $L^\infty(0, 2\pi)$. Since $\sup_n \|\sigma^n \hat{f}\|_{\hat{L}^\infty} = \|\hat{f}\|_{\hat{L}^\infty}$, [12, p. 137], we have

$$\begin{aligned}\sup_n \|\sigma^n \hat{f}\|_\alpha &= \sup_n \max\left\{ \|\sigma^n \hat{f}\|_{\hat{L}^\infty}, \sup_{h\neq 0} \frac{\|\sigma^n \hat{f}_h - \sigma^n \hat{f}\|_{\hat{L}^\infty}}{|h|^\alpha} \right\} \\ &= \max\left\{ \sup_n \|\sigma^n \hat{f}\|_{\hat{L}^\infty}, \sup_{h\neq 0} \sup_n \frac{\|\sigma^n (\hat{f}_h - \hat{f})\|_{\hat{L}^\infty}}{|h|^\alpha} \right\} \\ &= \|\hat{f}\|_\alpha .\end{aligned}$$

Hence $(\widehat{\text{lip}}\ \alpha)_{\sigma B} = \widehat{\text{Lip}}\ \alpha$. De Leeuw [2] has shown that $(\text{lip}\ \alpha)''$ is Lip α. By 3.6, $(\widehat{\text{lip}}\ \alpha)^\phi$ is a σK-space. □

4.2. REMARK. It follows from 3.5 that lip α also has a Cesàro basis. For TK-spaces, the β_T- and γ_T- duals are equal [1, Theorem 5]. Therefore, it follows that $\widehat{\text{Lip}}\ \alpha = ((\widehat{\text{lip}}\ \alpha)^\phi)^{\beta_\sigma} = (\widehat{\text{lip}}\ \alpha)_{F\sigma K}$ [1, Theorem 2].

The following characterization of reflexive Banach spaces with Toeplitz bases implies Theorems 7 and 8 of [3] and Theorem 3′ of [8].

4.3. THEOREM. *Let E be a BK-space and a TK-space. E is reflexive if and only if $E = (E^{\beta_T})^{\beta_T}$ and E^ϕ is a TK-space.*

PROOF. If E is a reflexive TK-space, then $E'' \cong (E^\phi)^\phi = E$. By 3.2, E^ϕ is an AD-space. By 3.5, E^ϕ is also a TB-space. Hence E^ϕ is a TK-space [11, Satz 4]. By [1, Theorem 5], $E = (E^{\gamma_T})^{\gamma_T} = (E^{\beta_T})^{\beta_T}$. Conversely, if E and E^ϕ are

TK-spaces, then by 3.2, 3.3, and [1, Theorem 5],

$$E'' \cong (E^{\phi})^{\phi} = E_{TB} = (E^{\gamma_T})^{\gamma_T} = (E^{\beta_T})^{\beta_T}.$$

If also $E = (E^{\beta_T})^{\beta_T}$, then $E'' \cong (E^{\phi})^{\phi} = E$. □

4.4. REMARK. It is clear from the proof that the condition $E = (E^{\beta_T})^{\beta_T}$ in Theorem 4.3 can be replaced by $E = (E^{\gamma_T})^{\gamma_T}$ or by $E = E_{TB}$. Also, by 3.5, the condition *E^{ϕ} is a TK-space* can be replaced by *E^{ϕ} is an AD-space.*

Let L, L^{∞}, C, and M be the Banach spaces of the 2π-periodic, integrable, essentially bounded measurable, continuous functions and bounded Borel measures, respectively. If E is one of these spaces, we define

$$\hat{E}_c = \left\{x \in \omega\colon \sum_{k=1}^{\infty} x_k \cos kt \text{ is the Fourier series of an } f \in E\right\}$$

and

$$\hat{E}_s = \left\{x \in \omega\colon \sum_{k=1}^{\infty} x_k \sin kt \text{ is the Fourier series of an } f \in E\right\}.$$

The spaces $\hat{E}_c$ and $\hat{E}_s$ are BK-spaces under the norm $\|x\| = \|f\|_E$ if f is the generating function (measure) of x. The theorem of F. and M. Riesz [12, p. 285] can be written $\hat{M}_c \cap \hat{M}_s = \hat{L}_c \cap \hat{L}_s$. Goes [4, Satz 1.22] has given characterizations of this equation in terms of the condition $F\sigma K$. Using 3.6, we easily obtain the following.

4.5. THEOREM. *The second dual of the BK-space $\hat{C}_c + \hat{C}_s$ can be identified with $\hat{L}_c^{\infty} + \hat{L}_s^{\infty}$ and this is an equivalent form of the F. and M. Riesz Theorem.*

PROOF. Let $E = \hat{C}_c + \hat{C}_s$. By Fejér's Theorem and [4, Satz 1.12], E is a σK-space. Thus

$$E' \cong E^{\varphi} = E^{\beta_\sigma} = (\hat{C}_c)^{\beta_\sigma} \cap (\hat{C}_s)^{\beta_\sigma} = \hat{M}_c \cap \hat{M}_s$$

and $E_{\sigma B} = \hat{L}_c^{\infty} + \hat{L}_s^{\infty}$ [4, Beispiele 1.19(b)]. By 3.6, $E'' \cong (E^{\phi})^{\phi} = E_{\sigma B} = \hat{L}_c^{\infty} + \hat{L}_s^{\infty}$ if and only if $E^{\phi} = \hat{M}_c \cap \hat{M}_s$ is a σK-space. But since $\hat{M}_c \cap \hat{M}_s$ is a σK-space if and only if

$$\hat{M}_c \cap \hat{M}_s = (\hat{M}_c \cap \hat{M}_s)_{\sigma K} = (\hat{M}_c)_{\sigma K} \cap (\hat{M}_s)_{\sigma K} = \hat{L}_c \cap \hat{L}_s,$$

our statement is indeed equivalent to the F. and M. Riesz Theorem. □

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