PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

WEAKLY ERGODIC HOMEOMORPHISMS

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ABSTRACT. We discuss self-homeomorphisms whose only invariant real-valued continuous functions are constants. We investigate the structure and properties of such homeomorphisms and give various examples.

1. **Introduction.** It is well known that if (X, Σ, μ, T) is a dynamical system with measure algebra (X, Σ, μ) and measure-preserving transformation T, then T is ergodic if and only if the only invariant measurable functions are constants, i.e. $f(Tx) = f(x)$ for f measurable implies that f is constant a.e. It is natural to ask if a topological analogue can be formulated. If X is a topological space and T is a self-homeomorphism of X, then is T ergodic if and only if $f(Tx) = f(x) \Rightarrow f$ is constant for any continuous real-valued function f on X? Many simple examples show that this is not the case and to characterise ergodicity we have to consider the ring of functions continuous on comeager sets [3]. Nevertheless the above property characterises a type of "almost" ergodic behaviour and the purpose of this paper is to begin the investigation of the structure, occurrence and properties of such homeomorphisms.

2. **Weak ergodicity.** Throughout this section (X, T) will be a cascade with compact, metric phase space X and self-homeomorphism T. $C(X)$ will be the ring of continuous real-valued functions on X and a function $f \in C(X)$ will be called *invariant* if $f(Tx) = f(x)$ for any $x \in X$.

DEFINITION 2.1. T is called *weakly ergodic* if $f \in C(X)$ and f invariant implies that f is a constant function.

REMARK. Every ergodic self-homeomorphism is weakly ergodic since there is an $x \in X$ with dense orbit under T, i.e. for some x the set $O_T(x) = \bigcup_{-\infty}^{\infty}\{T^i x\}$ is dense in X. Thus if f is invariant, $f|O_T(x)$ is a constant and so f is constant on X. The converse is false (see Example 1).

We now investigate the orbit structure of these homeomorphisms.

DEFINITION 2.2. A nonempty set $A \subset X$ is called *weakly minimal* in X if A is closed, A intersects no orbit closure of any point outside A and A is *minimal* with respect to this property. We call (X, T) *weakly minimal* if X is itself a weakly minimal set in X.

PROPOSITION 2.3. (a) *Every weakly minimal set is invariant.*
(b) *Every cascade has a weakly minimal set.*

Received by the editors May 17, 1975 and, in revised form, November 1, 1975.
AMS (MOS) subject classifications (1970). Primary 54H20.
Key words and phrases. Weakly ergodic, weakly minimal, ergodic.
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PROOF. (a) is clear. For (b) let $\mathcal{Q} = \{A \subset X;\ A$ is closed, $A \neq \varnothing$, and A intersects no orbit closure of a point outside $A\}$. As $X \in \mathcal{Q}$, $\mathcal{Q} \neq \varnothing$. Partially order $\mathcal{Q}$ by inclusion, let $\{A_\alpha\}$ be a chain in $\mathcal{Q}$ and consider $A = \bigcap A_\alpha$. A is closed and nonempty. Let $x \in X$ such that $\overline{O_T(x)} \cap A \neq \varnothing$. Then $\overline{O_T(x)} \cap A_\alpha \neq \varnothing$ for each α and so $x \in A_\alpha$ for each α. Thus $x \in A$. Hence $A \in \mathcal{Q}$. Zorn's lemma gives the result.

The hierarchy is as follows:

PROPOSITION 2.4. (a) (X, T) *ergodic implies* (X, T) *is weakly minimal.*
(b) (X, T) *weakly minimal implies* (X, T) *is weakly ergodic.*

PROOF. (a) If (X, T) is ergodic, it has a point x with a dense orbit. Thus every weakly minimal set must contain $O_T(x)$. Thus X itself is weakly minimal.

(b) Now suppose (X, T) is weakly minimal and let $f \in C(X)$ and invariant. Choose $x \in X$ and consider $f^{-1}(f(x))$. This set is closed and invariant since f is invariant so $(f^{-1}(f(x)), T)$ forms a cascade which has a weakly minimal set A by Proposition 2.3. Now let $z \in X - A$ such that $\overline{O_T(z)} \cap A \neq \varnothing$. Since A is weakly minimal in $f^{-1}(f(x)), f(z) \neq f(x)$. Let $T^{n_i}z \to a$ where $a \in A$. Then $f(T^{n_i}z) \to f(a)$ and so $f(z) = f(T^{n_i}z) = f(a) = f(x)$ which is a contradiction. Thus A is weakly minimal in X and so $A = X$. Thus $f^{-1}(f(x)) = X$ and so f is a constant function.

REMARK. Obviously a minimal *cascade* is weakly minimal, but a minimal *set* will not, in general, be weakly minimal.

EXAMPLE 1. To clarify the various concepts we examine the position when $X = [0, 1]$. First, there are no ergodic self-homeomorphisms. The weakly minimal self-homeomorphisms are those which have only a finite number of periodic points. For suppose T has a finite number of fixed points $x_1 < x_2 < \cdots < x_n$. Note that if (X, T^2) is weakly minimal so is (X, T), so we can assume T is increasing. Suppose A is a proper weakly minimal set in $[0, 1]$. Clearly $\overline{\mathcal{C}A}$ contains one of the fixed points, x_i say. But then $x_i \in A$ and any point in $\mathcal{C}A \cap [x_{i-1}, x_i]$ has x_i in its orbit closure.

Conversely, if T has an infinite number of fixed points, we can find a sequence of fixed points increasing (say) to a fixed point x. Now $[x, 1]$ (which may be $\{1\}$) contains a weakly minimal set as any $y < x$ falls into some interval $[x_r, x_{r+1}]$ and so $\overline{O_T(y)} \subset [x_r, x_{r+1}]$ and does not intersect $[x, 1]$. The last argument shows that a map with fixed points $\{1/n;\ n$ integer$\} \cup \{0\}$ will be weakly ergodic but not weakly minimal while Urysohn's theorem shows that a map whose set of fixed points has interior will not be weakly ergodic. This is not a necessary condition however; a homeomorphism which has the Cantor 1/3 set as its set of fixed points will not be weakly ergodic since there is a continuous real-valued function which is constant on each interval left out of $[0, 1]$ to give the Cantor set.

DEFINITION 2.5. Given $x \in X$ we call a closed set $A \subset X$ a *weakly minimal set for* x if $x \in A$, A intersects no orbit closure of a point outside A and A is minimal with respect to this property.

LEMMA 2.6. *There is a unique weakly minimal set for each* $x \in X$ *which we denote by* A_x.

PROOF. The existence follows as for 2.3(b). If A and B are both weakly minimal sets for x, then so is $A \cap B$ and thus $A = A \cap B = B$.

Clearly we also have

PROPOSITION 2.7. *(X, T) is weakly minimal if and only if $A_x = X$ for each $x \in X$.*

LEMMA 2.8. *There is a collection $\mathcal{Q}$ of closed, disjoint invariant sets covering X such that no refinement of $\mathcal{Q}$ has this property. Further, this cover is unique. The element of $\mathcal{Q}$ containing x is denoted by A^x. We call $\mathcal{Q}$ the weakly ergodic decomposition.*

PROOF. Let $\Gamma = \{\mathcal{Q};\ \mathcal{Q}$ is a closed, disjoint cover of X consisting of invariant sets$\}$. Index Γ by $\Gamma = \{\mathcal{Q}_\beta; \beta \in \mathfrak{B}\}$. Γ is nonempty as $\{X\} \in \Gamma$. For each $x \in X$ let

$$A^x = \cap\{A_\alpha; A_\alpha \in \mathcal{Q}_\alpha \text{ and } x \in A_\alpha\}.$$

It is easy to check that $\{A^x; x \in X\}$ is the required cover.

Justifying the term *weakly ergodic decomposition*:

PROPOSITION 2.9. *(A^x, T) is weakly ergodic for each $x \in X$.*

PROOF. Let f be a continuous real-valued invariant function on A^x. Consider $\{f^{-1}(\alpha);\ \alpha$ real$\}$. This is a closed, disjoint, invariant cover and so $\{f^{-1}(\alpha);\ \alpha$ real$\} \cup \{A^y; A^y \neq A^x\}$ is a cover in Γ. But now $A^x \subset f^{-1}(f(x)) \subset A^x$, so f is constant.

EXAMPLE 2. A^x need not be A_x and it is not necessary that each $A^x = X$ for weak ergodicity. Let X be the subspace of R^2 consisting of $[0, 1] \times [0, 1]$ with a set of nonoverlapping arcs starting at $(1, y)$ and finishing at $(0, 2 - y)$. We define T on X by $T(x,y) = (x, y^{2-x})$ for $0 \leqslant x < 1$, $0 \leqslant y \leqslant 1$ and similarly for the arcs so that the arcs have fixed endpoints and each arc is a weakly minimal set in X. Clearly for each (x,y) on an arc, $A_{(x,y)} = A^{(x,y)} =$ arc, and for each (x,y) in the unit square, $A_{(x,y)} = A^{(x,y)} = \{(x, a); 0 \leqslant a \leqslant 1\}$. However (X, T) is weakly ergodic. We can check this directly by following orbits or apply the next theorem.

We now characterise weak ergodicity in terms of orbit decompositions. To simplify notation, if $\mathfrak{D}$ is a disjoint, closed cover, we denote by D the decomposition space of $\mathfrak{D}$, i.e. the space whose points are the elements of $\mathfrak{D}$ and a set is open in D if the union of its elements is open in X. Denote by D^* the universal Hausdorff space associated with D, i.e. the Hausdorff space such that any continuous map f from D to a Hausdorff space can be lifted to a continuous map in D^* [1, p. 140].

THEOREM 2.10. *Let (X, T) be a cascade and D the decomposition space of the weakly ergodic decomposition of X. Then (X, T) is weakly ergodic if and only if D^* is one point.*

PROOF. Let p be the projection map from X onto D and let ϕ be the universal map from D onto D^*. Then $\phi \circ p$ is a continuous map from X onto D^* and so, as D^* is Hausdorff, it is compact and so normal. Suppose D^* has more than one point. Then there is a continuous real-valued function f on D^* which is not constant. Thus $f \circ \phi \circ p$ is continuous and not constant on X. By the construction of $\mathfrak{D}$, p is invariant and so $f \circ \phi \circ p$ is invariant. Thus (X, T) is not weakly ergodic.

Conversely, suppose (X, T) is not weakly ergodic. Let f be a continuous invariant function which is not constant. We can define a function F on D by $F(A^x) = f(x)$ since, by Proposition 2.9, $f|A^x$ must be constant. Clearly F is not constant. Finally by the universal property there is a function F^* on D^* such that $F^* \circ \phi = F$ and so F^* is not constant. Thus D^* has more than one point.

Example 3. (a) In Example 2, $D = \{(x,0); 0 \leqslant x < 1\} \cup \{(1,y); 0 \leqslant y < 1\} \cup \{(1,1)\}$ with the topology

$$N_\varepsilon(x,0) = (x - \varepsilon, x + \varepsilon) \times \{0\} \cup \{1\} \times (1 - \varepsilon, 1) \cup (1 - \varepsilon, 1) \times \{0\},$$

etc. Since this topology has *no* disjoint neighbourhoods, D^* is one point.

(b) If X is the torus with the product rotation $T \times T$ where T is a rotation incommensurable with π of the unit circle, then D is just the unit circle and $D^* = D$.

Remark. We can also describe D^* directly from the function algebra. It is easy to check that D^* is actually the structure space of $\{f \in C(X); f \circ T = f\}$. This gives us another proof of 2.10.

3. **Inheritance properties.** Ergodic cascades are not well behaved under inheritance properties. Factors of ergodic cascades are ergodic, but Example 3(b) above shows that products need not be ergodic (or even weakly ergodic) and we can easily construct examples in which no power of an ergodic homeomorphism (except -1) is ergodic. In this section we show that under certain conditions weakly ergodic and weakly minimal cascades are well behaved under products and powers. For simplicity a *clopen set invariant under* T will be a nonempty open and closed proper subset of the phase space invariant under the homeomorphism T.

Proposition 3.1. *Factors of weakly ergodic cascades are weakly ergodic and factors of weakly minimal cascades are weakly minimal. Thus weak ergodicity and weak minimality are isomorphism invariants.*

Proof. Follows immediately from the definitions.

Lemma 3.2. *Let (X, T) be a weakly ergodic cascade and n a nonzero integer. Then the space D^* for the cascade (X, T^n) has at most n distinct points.*

Proof. Let $\{A^x; x \in X\}$ be the weakly ergodic decomposition of (X, T) and $\{B^x; x \in X\}$ be the weakly ergodic decomposition of (X, T^n). For each A^x, there is a B^y such that $A^x = B^y \cup TB^y \cup \cdots \cup T^{n-1}B^y$. For any integer i, $\{T^iB^x; x \in X\}$ is a closed, disjoint cover invariant under T^n and so for each TB^x there is a B^y with $B^y \subset TB^x$. But then $T^{n-1}B^y \subset B^x$ and so $B^y = TB^x$. Thus T induces a periodic homeomorphism on the decomposition space for (X, T^n) and so it induces a homeomorphism T^* on D^* and T^* is also periodic with period at most n.

Now let $R \subset D^* \times D^*$ be the equivalence relation defined by $(x,y) \in R$ if $y = T^{*i}x$ for some integer i. If $(x,y) \in \mathrm{cl}\,(R)$, then for some k every neighbourhood of (x,y) contains a point of the form $(a, T^k a)$ so $y = T^k x$ and $(x,y) \in R$. Thus R is closed and D^*/R is Hausdorff. If E is the decomposition space for (X, T) then it is easy to check that the map from E onto D^*/R which takes A^x onto the equivalence class of the set $\{B^y, TB^y, \ldots, T^{n-1}B^y\}$ in D^*/R is a well-defined continuous surjection. But then by the universal property there is a map from E^* onto D^*/R which means that D^*/R is one point and so D^* can have at most n points.

THEOREM 3.3. *Let (X, T) be a weakly ergodic cascade and n be a nonzero integer. Then (X, T^n) is weakly ergodic if and only if there is no clopen set invariant under T^n. This statement is also true with weakly minimal replacing weakly ergodic.*

PROOF. Necessity is obvious in each case. Suppose that (X, T^n) is not weakly ergodic. By Lemma 3.2, D^* for (X, T^n) contains between 2 and n points and, as D^* is Hausdorff, each point is open and closed. Thus if p is the projection to the decomposition space and ϕ is the universal map, for any $x \in D^*$, $p^{-1}\phi^{-1}(x)$ is a clopen set in X invariant under T^n. Next suppose (X, T^n) is not weakly minimal and let $x \in X$ such that $A_x \neq X$. Now $T^i A_x$ is weakly minimal in X for any integer i and so $T^i A_x \cap T^j A_x = \varnothing$ or $T^i A_x = T^j A_x$. Consider $A = A_x \cup TA_x \cup \cdots \cup T^{n-1}A_x$. A is closed and invariant under T. If $A \neq X$, there is a $y \in X - A$ such that $\overline{O_T(y)} \cap A \neq \varnothing$ as X is weakly minimal. But then $\overline{O_T(y)} \cap A_x \neq \varnothing$ and so one of $O_{T^n}(y), \ldots, O_{T^n}(T^{n-1}y)$ has a closure point in A_x which is a contradiction. Thus $A = X$ and so A_x is open and closed.

COROLLARY 3.4. *Let (X, T) be a weakly ergodic cascade.*

(a) *If X is connected, then (X, T^n) is weakly ergodic for any $n \neq 0$.*

(b) *If T has a fixed point, then (X, T^n) is weakly ergodic for any $n \neq 0$.*

These statements are also true with weakly minimal replacing weakly ergodic.

PROOF. (a) is obvious. For (b), suppose that A is a clopen set invariant under T^n for some n and p is a fixed point of T. Since $\bigcup_{i=1}^n T^i A$ and $\bigcap_{i=1}^n T^i A$ are closed and open sets invariant under T, the first set is X so $p \in A$, and the second is empty so $p \notin A$.

We can also obtain a result for products.

THEOREM 3.5. *Let (X, T) be a weakly minimal cascade, (Y, S) be a weakly ergodic cascade and suppose that T has a fixed point. Then $(X \times Y, T \times S)$ is weakly ergodic.*

PROOF. Let f be an invariant function in $C(X \times Y)$. The function $g(y) = f(p,y)$, where p is a fixed point of T, is an invariant function in $C(Y)$ and so a constant c, say. Let $\mathscr{Q} = \{A \subset X; f(x,y) = c$ for all $x \in A$ and all $y \in Y\}$. $\{p\} \in \mathscr{Q}$ so $\mathscr{Q} \neq \varnothing$. Partially order $\mathscr{Q}$ by inclusion and select a maximal element M. Clearly M is closed and invariant under T so if $M \neq X$ there must be an $x \in X$ such that $\overline{O_T(x)} \cap M \neq \varnothing$ since X is weakly minimal. Thus for any $y \in Y$ we can find integers n_i such that $(T^{n_i}x, T^{n_i}y) \to (a,b)$ where $a \in M$ and $b \in Y$. Thus $f(T^{n_i}x, T^{n_i}y) = f(x,y) = f(a,b)$

$= c$. But now $M \cup \{x\} \in \mathscr{Q}$, which is a contradiction, and so $M = X$. Thus f is a constant function.

COROLLARY 3.6 . *Let* (X, T) *be weakly minimal and* (Y, S) *be weakly ergodic, and let* T *have a periodic point of period* n. *Suppose that there is no clopen set invariant under* T^n *and no clopen set invariant under* S^n. *Then* $(X \times Y, T \times S)$ *is weakly ergodic.*

PROOF. By 3.3, (X, T^n) is weakly minimal and (Y, S^n) is weakly ergodic. By 3.5, $(X \times Y, T^n \times S^n) = (X \times Y, (T \times S)^n)$ is weakly ergodic and so $(X \times Y, T \times S)$ is weakly ergodic.

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