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PRINCIPAL CONGRUENCES OF p-ALGEBRAS AND DOUBLE p-ALGEBRAS

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ABSTRACT. Principal congruence of pseudocomplemented lattices (= p-algebras) and of double pseudocomplemented lattices (= double p-algebras), i.e. pseudocomplemented and dual pseudocomplemented ones, are characterized.

1. **Introduction.** Recently H. Lakser [7] proved that every principal congruence of a distributive p-algebra is a join of two principal lattice congruences. We shall extend this result to all p-algebras (Theorem 1). The situation changes radically if one examines the double p-algebras. By Theorem 2, every principal congruence of a double p-algebra is a join of countably many principal lattice congruences. There exists even a distributive double p-algebra having a principal congruence which cannot be represented as a join of finite principal lattice congruences (Lemmas 3, 4 and Example). In Theorem 3 we give a necessary and sufficient condition in order that every principal congruence of a double p-algebra be a join of finite principal lattice congruences.

2. **Preliminaries.** A universal algebra $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ of type $\langle 2,2,1,0,0\rangle$ is called a *p-algebra* iff $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice such that for every $a \in L$ the element $a^* \in L$ is the *pseudocomplement* of a, i.e. $x \leqslant a^*$ iff $a \wedge x = 0$. A universal algebra $\langle L; \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ is called a *double* p-algebra iff $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ is a p-algebra and $\langle L; \vee, \wedge, {}^+, 0, 1\rangle$ is a *dual* p-algebra ($x \geqslant a^+$ iff $x \vee a = 1$). The standard results on p-algebras may be found in [3].

For a p-algebra L, define the set $B(L) = \{x \in L: x = x^{**}\}$ of *closed* elements. The partial ordering of L partially orders $B(L)$ and makes the latter into a Boolean algebra $\langle B(L); \cup, \wedge, {}^*, 0, 1\rangle$ for which $a \cup b = (a \vee b)^{**}$ holds.

For any pair $a, b \in L$ in a p-, dual p-, or double p-algebra L, $\theta(a,b)$ denotes the principal congruence relation generated by a, b, i.e. the least congruence relation θ of this algebra for which $a \equiv b(\theta)$ is true. Clearly

$$\theta(a,b) = \theta(a \wedge b, a \vee b);$$

thus we need only characterize $\theta(a,b)$ for comparable a, b. We denote by

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$\theta_{\mathrm{Lat}}(a,b)$ the principal lattice congruence generated by a, b; $\theta_{\mathrm{Lat}}(a,b)$ has the substitution property for $\wedge$ and $\vee$, but not necessarily for * or $^+$.

For the definition of a unary algebraic function see [2]. By a unary lattice function we mean such a unary algebraic function which can be obtained from a lattice polynomial (see also [3]).

3. Principal congruences of p-algebras.

LEMMA 1. *Let L be a p-algebra. Let a, $b \in L$ and $a \leqslant b$. If $p(x)$ is a unary lattice function on L then the following identities hold*:

(i) $p(a)^* \wedge a^{**} = p(b)^* \wedge a^{**}$;

(ii) $p(a)^* \wedge b^* = p(b)^* \wedge b^*$.

PROOF. We proceed by induction on the rank of the lattice polynomial $r(x_0, \ldots, x_{n-1})$, where $r(x, c_1, \ldots, c_{n-1}) = p(x)$, $c_1, \ldots, c_{n-1} \in L$. If $p(x)$ is the identity or constant function (i) and (ii) hold trivially. Examine $p(x) = q(x) \wedge t(x)$, $q(x)$ and $t(x)$ satisfying (i). Then

$$p(a)^* \wedge a^{**} = [q(a) \wedge t(a)]^* \wedge a^{**} = (q(a)^* \wedge a^{**}) \cup (t(a)^* \wedge a^{**})$$
$$= (q(b)^* \wedge a^{**}) \cup (t(b)^* \wedge a^{**}) = p(b)^* \wedge a^{**}.$$

Similarly, if $p(x) = q(x) \vee t(x)$ and $q(x)$ and $t(x)$ satisfy (i), then

$$p(a)^* \wedge a^{**} = [q(a) \vee t(a)]^* \wedge a^{**} = q(a)^* \wedge t(a)^* \wedge a^{**}$$
$$= q(b)^* \wedge t(b)^* \wedge a^{**} = p(b)^* \wedge a^{**}.$$

Thus we have proved property (i). The proof of (ii) is similar.

LEMMA 2. *Let L be a lattice with* 1 *and let $d \in L$. We define a binary relation θ_d on L in the following way*:

$$x \equiv y(\theta_d) \quad \textit{iff } x \wedge d = y \wedge d.$$

Then $\theta_d \leqslant \theta_{\mathrm{Lat}}(d, 1)$.

The proof is straightforward.

THEOREM 1. *Let L be a p-algebra, let a, $b \in L$ and let $a \leqslant b$. Then*

$$\theta(a,b) = \theta_{\mathrm{Lat}}(a,b) \vee \theta_{\mathrm{Lat}}((a^* \wedge b)^*, 1). \tag{1}$$

PROOF. Let θ denote the lattice congruence on the right-hand side of (1). First we show that θ has the substitution property with respect to the operation *. Let $x \equiv y(\theta)$. Then there is a sequence $x = z_0, \ldots, z_n = y$ of elements of L and a sequence $p_0, \ldots, p_{n-1}$ of unary lattice functions such that

(I) $\{z_i, z_{i+1}\} = \{p_i(a), p_i(b)\}$ or

(II) $\{z_i, z_{i+1}\} = \{p_i((a^* \wedge b)^*), p_i(1)\}$ for any $i = 0, 1, \ldots, n-1$ holds.

Consider case (I). By Lemma 1, we have

$$z_i^* \wedge a^{**} = z_{i+1}^* \wedge a^{**}, \qquad z_i^* \wedge b^* = z_{i+1}^* \wedge b^*.$$

Since $B(L)$ is a Boolean algebra, we get

$$z_i^* \wedge (a^{**} \cup b^*) = z_{i+1}^* \wedge (a^{**} \cup b^*).$$

By Lemma 2, the last identity implies $z_i^* \equiv z_{i+1}^*(\theta_{\mathrm{Lat}}((a^* \wedge b)^*, 1))$, because $a^{**} \cup b^* = (a^* \wedge b)^*$. In case (II) we obtain

$$z_i^* \wedge (a^* \wedge b)^* = z_{i+1}^* \wedge (a^* \wedge b)^*,$$

by Lemma 1(i), bearing in mind $(a^* \wedge b)^* \in B(L)$. This implies $z_i^* \equiv z_{i+1}^*(\theta_{\mathrm{Lat}}((a^* \wedge b)^*, 1))$ by Lemma 2. So, $x^* \equiv y^*(\theta)$ and θ is a *-congruence of L. Evidently $\theta(a,b) \leqslant \theta$. Conversely, $a \equiv b(\theta(a,b))$ yields $a^* \wedge b \equiv 0(\theta(a,b))$, and hence $(a^* \wedge b)^* \equiv 1(\theta(a,b))$. Thus, $\theta(a,b) \geqslant \theta$. Concluding, $\theta(a,b) = \theta$.

COROLLARY 1. *Let L be a p-algebra. Then $\theta(a,1) = \theta_{\mathrm{Lat}}(a,1)$ for every $a \in L$.*

COROLLARY 2. *Let L be a dual p-algebra, let a, $b \in L$ and let $a \leqslant b$. Then*

$$\theta(a,b) = \theta_{\mathrm{Lat}}(a,b) \vee \theta_{\mathrm{Lat}}((a \vee b^+)^+, 0). \tag{2}$$

COROLLARY 3. *Let L be a dual p-algebra. Then $\theta(0,a) = \theta_{\mathrm{Lat}}(0,a)$ for every $a \in L$.*

REMARK 1. The analogue of Theorem 1 is also valid for the pseudocomplemented semilattices. (The proof of Theorem 1 is based on the fact that $B(L)$ is a Boolean algebra.)

REMARK 2. Theorem 1 was proved in [7] for the distributive p-algebras. In [4], an equivalent version of Theorem 1 has been proved for the modular S-algebras.

4. **Principal congruences of double p-algebras.** Let L be a double p-algebra, let $x \in L$. We define $x^{n(+*)} \in L$ in the following way: $x^{1(+*)} = x^{+*}$, $x^{(k+1)(+*)} = x^{k(+*)+*}$ for every $k \geqslant 1$. Similarly we define $x^{n(*+)} \in L$. Since $a^* \vee a^{*+} = 1$ implies $a^{**} \wedge a^{*+*} = 0$, we obtain $a^{*+*} \leqslant a^*$ in L. Therefore,

$$x^* \geqslant x^{*+*} \geqslant \cdots \geqslant x^{*n(+*)} \geqslant \cdots \tag{3}$$

in L. Dually we have

$$y^+ \leqslant y^{+*+} \leqslant \cdots \leqslant y^{+n(*+)} \leqslant \cdots \tag{4}$$

for any $y \in L$.

THEOREM 2. *Let L be a double p-algebra, let a, $b \in L$ and let $a \leqslant b$. Then*

$$\begin{aligned} \theta(a,b) &= \theta_{\mathrm{Lat}}(a,b) \\ &\quad \vee \bigvee_{n \geqslant 0} [\theta_{\mathrm{Lat}}((a^* \wedge b)^{*n(+*)}, 1) \vee \theta_{\mathrm{Lat}}(0, (a \vee b^+)^{+n(*+)})]. \end{aligned} \tag{5}$$

PROOF. Let θ denote the lattice congruence on the right-hand side of (5). It is a routine to show that $\theta(a,b) \geqslant \theta$. To conclude the proof we need only to show that θ has the substitution property with respect to the operations * and

$^+$. First we prove that θ is a *-congruence. Let $x \equiv y(\theta)$. Then there is a sequence $x = z_0, \ldots, z_n = y$ of elements of L and congruences $\theta_0, \ldots, \theta_{n-1}$ such that $z_i \equiv z_{i+1}(\theta_i)$ where

(1) $\theta_i = \theta_{\text{Lat}}(a, b)$ or

(2) $\theta_i = \theta_{\text{Lat}}((a^* \wedge b)^{*k(+*)}, 1)$ for some $k \geqslant 0$ or

(3) $\theta_i = \theta_{\text{Lat}}(0, (a \vee b^+)^{+m(*+)})$ for some $m \geqslant 0$, and for any $i = 0, \ldots, n-1$.

(1) $z_i \equiv z_{i+1}(\theta_{\text{Lat}}(a, b))$ implies

$$z_i \equiv z_{i+1}(\theta_{\text{Lat}}(a, b) \vee \theta_{\text{Lat}}((a^* \wedge b)^*, 1)),$$

by Theorem 1. Therefore $z_i^* \equiv z_{i+1}^*(\theta)$.

(2) $z_i \equiv z_{i+1}(\theta_{\text{Lat}}((a^* \wedge b)^{*k(+*)}, 1))$ implies

$$z_i^* \equiv z_{i+1}^*(\theta_{\text{Lat}}((a^* \wedge b)^{*k(+*)}, 1)),$$

by Corollary 1 to Theorem 1. Therefore $z_i^* \equiv z_{i+1}^*(\theta)$.

(3) $z_i \equiv z_{i+1}(\theta_{\text{Lat}}(0, (a \vee b^+)^{+m(*+)}))$ implies

$$z_i^* \equiv z_{i+1}^*(\theta_{\text{Lat}}(0, (a \vee b^+)^{+m(*+)}) \vee \theta_{\text{Lat}}((a \vee b^+)^{+m(*+)*}, 1)),$$

by Theorem 1. Since $(a \vee b^+)^{+(m+1)(*+)} \vee (a \vee b^+)^{+m(*+)*} = 1$, we have

$$\theta_{\text{Lat}}(0, (a \vee b^+)^{+(m+1)(*+)}) \geqslant \theta_{\text{Lat}}((a \vee b^+)^{+m(*+)*}, 1).$$

Therefore $z_i^* \equiv z_{i+1}^*(\theta)$.

Thus, $z_i^* \equiv z_{i+1}^*(\theta)$ for any $i = 0, \ldots, n-1$, and we have proved $x^* \equiv y^*(\theta)$, i.e. θ is a *-congruence of L. Using Corollaries 2 and 3 to Theorem 1 one can similarly prove that θ is a $^+$-congruence of L, and so the proof is complete.

COROLLARY 1. *Let L be a double p-algebra, $a \in L$. Then*

$$\theta(a, 1) = \bigvee_{n \geqslant 0} \theta_{\text{Lat}}(0, a^{+n(*+)}); \tag{6}$$

$$\theta(0, a) = \bigvee_{n \geqslant 0} \theta_{\text{Lat}}(a^{*n(+*)}, 1). \tag{7}$$

PROOF. We know that $\theta_{\text{Lat}}(a, 1) \leqslant \theta_{\text{Lat}}(0, a^+)$ and $\theta_{\text{Lat}}(a^{**n(+*)}, 1) \leqslant \theta_{\text{Lat}}(a^{n(+*)}, 1) \leqslant \theta_{\text{Lat}}(0, a^{+n(*+)})$ is true. Hence, by Theorem 2, we have (6). By dual arguments we can prove (7).

COROLLARY 2. *Let L be a double p-algebra in which the chains* (3) *and* (4) *are finite for every $x, y \in L$. Let $a, b \in L$ with $a \leqslant b$. Let m be the least number with the property $(a^* \wedge b)^{*m(+*)} = (a^* \wedge b)^{*(m+1)(+*)}$ and $(a \vee b^+)^{+m(*+)} = (a \vee b^+)^{+(m+1)(*+)}$. Then*

$$\theta(a, b) = \theta_{\text{Lat}}(a, b) \vee \theta_{\text{Lat}}((a^* \wedge b)^{*m(+*)}, 1) \vee \theta_{\text{Lat}}(0, (a \vee b^+)^{+m(*+)}).$$

COROLLARY 3. *Let L be a distributive double p-algebra satisfying the identities*

$$x^{*m(+*)} = x^{*(m+1)(+*)}, \qquad x^{+m(*+)} = x^{+(m+1)(*+)}$$

for some $m \geqslant 0$. Let $a, b \in L$ with $a \leqslant b$. If $x, y \in L$, then $x \equiv y(\theta(a,b))$ iff

$$[x \wedge a \wedge (a^* \wedge b)^{*m(+*)}] \vee (a \vee b^+)^{+m(*+)}$$
$$= [y \wedge a \wedge (a^* \wedge b)^{*m(+*)}] \vee (a \vee b^+)^{+m(*+)}$$

and

$$[(x \vee b) \wedge (a^* \wedge b)^{*m(+*)}] \vee (a \vee b^+)^{+m(*+)}$$
$$= [(y \vee b) \wedge (a * \wedge b)^{*m(+*)}] \vee (a \vee b^+)^{+m(*+)}.$$

PROOF. The proof follows from Corollary 2 and the fact that in a bounded distributive lattice L, for any elements $a_1, b_1, a_2, b_2 \in L$ with $a_1 \leqslant b_1$, the following statement is true:

$$x \equiv y(\theta_{\mathrm{Lat}}(a_1, b_1) \vee \theta_{\mathrm{Lat}}(a_2, 1) \vee \theta_{\mathrm{Lat}}(0, b_2))$$

iff

$$(x \wedge a_1 \wedge a_2) \vee b_2 = (y \wedge a_1 \wedge a_2) \vee b_2$$

and

$$[(x \vee b_1) \wedge a_2] \vee b_2 = [(y \vee b_1) \wedge a_2] \vee b_2$$

hold (see [3]).

REMARK. Corollary 3 combined with the result of A. Day [1] says that the equational subclass of the class of all distributive double p-algebras determined by the identities from Corollary 3 enjoys the Congruence Extension Property. We note here that the whole class of distributive double p-algebras has CEP (see [5]). Corollary 3 solves partially the problem mentioned in [5].

5. **Counterexample.** In this part we shall construct a distributive double p-algebra having a principal congruence which cannot be represented as a join of finite principal lattice congruences.

LEMMA 3. *Let L be a double p-algebra. If $a \in L$ and $a^{*n(+*)} > a^{*(n+1)(+*)}$ ($a^{+n(*+)} < a^{+(n+1)(*+)}$) for every integer $n \geqslant 0$ then $\theta(0,a)$ ($\theta(a, 1)$) cannot be represented as a join of finite principal lattice congruences of L.*

PROOF. Let $a^{*n(+*)} > a^{*(n+1)(+*)}$ for any $n \geqslant 0$. Suppose to the contrary that $\theta(0,a)$ is a join of finite principal lattice congruences of L. Then $\theta(0,a)$ is a compact element of the lattice of all lattice congruences on L (cf. [2]). Therefore, by (3) and (6), there exists an integer $k \geqslant 0$ such that

$$\theta(0,a) = \bigvee_{n=0}^{k} \theta_{\mathrm{Lat}}(a^{*n(+*)}, 1) = \theta_{\mathrm{Lat}}(a^{*k(+*)}, 1).$$

Evidently $a^{*(k+1)(+*)} \equiv 1(\theta(0,a))$. On the other hand,

$$a^{*(k+1)(+*)} \not\equiv 1(\theta_{\mathrm{Lat}}(a^{*k(+*)}, 1))$$

(see [3]), a contradiction. The second part can be proved dually.

LEMMA 4. *Let B be a Boolean algebra, let φ: $B \to B$ be a $\{0, 1, \wedge\}$-homomorphism and let ψ: $B \to B$ be a $\{0, 1, \vee\}$-homomorphism such that $a\varphi\psi \leqslant a$ and $a\psi\varphi \geqslant a$ for every $a \in B$. Then the set $L = \{(a,b) \in B^2: a\varphi \geqslant b\}$ is a $\{0,1\}$-sublattice of B^2 and, moreover, L forms a distributive double p-algebra in which for $t = (a,b) \in L$,*

$$t^* = (a', a'\varphi), \qquad t^+ = (b'\psi, b')$$

is true.

For the proof see [6, Theorem 2].

EXAMPLE. Let B denote the Boolean algebra of all subsets of the set N of positive integers. Set

$A\varphi = \{x \in N: x \in A$ and $x + 1 \in A\}$ for every $A \in B$,

$A\psi = \{x \in N: x \in A$ or $x - 1 \in A\}$ for every $A \in B$.

It is easy to verify that φ is a $\{0, 1, \wedge\}$-homomorphism of B into B, ψ is a $\{0, 1, \vee\}$-homomorphism of B into B, both of which satisfy $A\varphi\psi \leqslant A$ and $A\psi\varphi \geqslant A$ for every $A \in B$. Let $L = \{(X, Y) \in B^2: X\varphi \geqslant Y\}$ be the distributive double p-algebra (see Lemma 4). Let $K_n = \{1, \dots, n\} \in B$. If we set $a = (N - K_1, N - K_1)$ and $b = a^+$ then $a^+ = b = (K_2, K_1)$, $a^{+*} = (N - K_2, N - K_2)$, $a^{+*+} = b^{*+} = (K_3, K_2)$. By induction it is easy to prove

$a^{+n(*+)} = (K_{n+2}, K_{n+1})$,

$b^{*n(+*)} = (N - K_{n+2}, N - K_{n+2})$.

Now we see that $b^{*n(+*)} > b^{*(n+1)(+*)}$ and $a^{+n(*+)} < a^{+(n+1)(*+)}$ for every integer $n \geqslant 0$. So, by Lemma 3, $\theta(a, 1)$ and $\theta(0, b)$ cannot be represented as a join of finite principal lattice congruences of L.

Concluding we obtain

THEOREM 3. *Let L be a double p-algebra. Let a, $b \in L$. Then the principal congruence $\theta(a, b)$ is a join of finite principal lattice congruences of L iff the chains* (3) *and* (4) *are finite for every x, $y \in L$.*

The proof follows from Corollary 2 to Theorem 2 and Lemma 3.

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