

## WHEN IS THE TENSOR PRODUCT OF ALGEBRAS LOCAL? .II

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**ABSTRACT.** In a recent paper, Sweedler gave necessary and sufficient conditions for the tensor product of two commutative algebras to be local. We study the noncommutative case. We show that his conditions are necessary in this case. The conditions are not in general, sufficient, and an algebraic condition must be replaced by local finiteness.

All rings are associative with unity. The Jacobson radical of a ring is denoted by  $J(R)$ .  $R$  is local if  $\bar{R} = R/J(R)$  is a division ring. All algebras and tensor products are over fields. An algebra is algebraic if each element generates a finite dimensional subalgebra. An algebra is locally finite if every finite subset generates a finite dimensional subalgebra.

In a recent paper, Sweedler gave necessary and sufficient conditions for the tensor product of two commutative algebras to be local.

**THEOREM 1** [4]. *Let  $A$  and  $B$  be commutative  $F$ -algebras. Then  $A \otimes_F B$  is local if and only if:*

- (1)  *$A$  and  $B$  are local,*
- (2)  *$\bar{A} \otimes_F \bar{B}$  is local,*
- (3) *either  $A$  or  $B$  is  $F$ -algebraic.*

In this paper we show that his conditions are necessary in the noncommutative case. For sufficiency, "algebraic" must be replaced by "locally finite" in (3).

**LEMMA 2.** *Let  $A$  and  $C$  be  $F$ -algebras such that  $A \otimes_F C$  is local. If  $B$  is a division subring of  $A$  containing  $F$ , then  $B \otimes_F C$  is local. Conversely, if  $A$  is a locally finite division ring such that for every finitely generated division subring  $B$  containing  $F$ ,  $B \otimes_F C$  is local, then  $A \otimes_F C$  is local.*

**PROOF.** Suppose  $\alpha \in B \otimes C$  is invertible in  $A \otimes C$ . Then it is invertible in  $B \otimes C$ . This can be proved by taking a left  $B$ -basis for  $A$ . Let  $\{\partial_k\}$  be such a basis with  $1 = \partial_1 \in \{\partial_k\}$ . Suppose that  $\alpha = \sum_i (b_i \otimes c_i)$  has inverse  $\beta = \sum_j (a_j \otimes c_j)$ . Then  $a_j = \sum b_{jk} \partial_k$ , hence

$$1 = \alpha\beta = \left[ \sum_i (b_i \otimes c_i) \right] \left[ \sum_j \left( \sum_k b_{jk} \partial_k \otimes c_j \right) \right] = \sum_{ij} \left( \sum_k b_i b_{jk} \partial_k \otimes c_i c_j \right).$$

By a basis argument, we obtain

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$$1 = \left[ \sum_i (b_i \otimes c_i) \right] \left[ \sum_j (b_{j1} \otimes c_j) \right].$$

Now let  $\alpha, \alpha' \in B \otimes C$  be two noninvertible elements. Then, if  $A \otimes C$  is local,  $\alpha + \alpha'$  is not invertible in  $A \otimes C$ , hence not invertible in  $B \otimes C$ . Thus if  $A \otimes C$  is local,  $B \otimes C$  is local. Conversely, let  $\alpha'$  and  $\alpha$  be two elements in  $A \otimes C$ . Then there is a finitely generated division subring (finitely generated as a division ring)  $B$  such that  $\alpha$  and  $\alpha'$  are in  $B \otimes C$ . If the two elements are not invertible in  $A \otimes B$ , then they are not invertible in  $B \otimes C$ ; hence, their sum is not invertible in  $B \otimes C$  if  $B \otimes C$  is local. This proves the second half of the lemma.

**THEOREM 3.** *Let  $A$  and  $B$  be  $F$ -algebras. If  $A \otimes_F B$  is local, then*

- (1)  *$A$  and  $B$  are local,*
- (2)  *$\bar{A} \otimes_F \bar{B}$  is local,*
- (3) *either  $A$  or  $B$  is  $F$ -algebraic.*

**PROOF.** Let  $L$  be the unique maximal right ideal of  $A \otimes B$ . If  $I$  is a proper right ideal of  $B$ , then  $A \otimes I$  is a right ideal of  $A \otimes B$ , hence is contained in  $L$ . Therefore  $B$  is local and  $J(A \otimes B) \cap B = J(B)$ . Similarly  $A$  is local. Since  $\bar{A} \otimes \bar{B}$  is a homomorphic image of  $A \otimes B$ , it is local. We now claim that either  $\bar{A}$  or  $\bar{B}$  is  $F$ -algebraic. If not, choose  $a \in \bar{A}$  and  $b \in \bar{B}$  transcendental over  $F$ . By the previous lemma,  $F(a) \otimes F(b)$  is local, contradicting Sweedler's theorem. Thus we may assume that  $\bar{B}$  is  $F$ -algebraic. If  $A$  is not algebraic, then  $J(A \otimes B) \cap B = J(B)$  is a nil ideal [1], whence  $B$  is algebraic.  $\square$

Parts of the above theorem can be generalized still further. If  $A \otimes_F B$  is semilocal (modulo the radical the algebra is artinian), then either  $A$  or  $B$  is algebraic [2].

**THEOREM 4.** *Let  $A$  and  $B$  be local  $F$ -algebras such that  $A$  is locally finite, and  $\bar{A} \otimes_F \bar{B}$  is local. Then  $A \otimes_F B$  is local.*

**PROOF.**  $J(A)$  is nil, hence  $J(A) \otimes B$  is a nil ideal of  $A \otimes B$ , and so it is contained in  $J(A \otimes B)$ . The problem thus reduces to showing that  $\bar{A} \otimes B$  is local. By Lemma 2, we may assume that  $\bar{A}$  is finite dimensional over  $F$ . But in this case  $\bar{A} \otimes J(B) \subseteq J(A \otimes B)$  [3, Theorem 16.3], and so it is sufficient to show that  $\bar{A} \otimes \bar{B}$  is local, which is one of the given conditions.  $\square$

Let  $A = F[[x, y]]$  be the power series ring in two noncommuting indeterminates.  $A$  is local and  $\bar{A} \cong F$ .

**THEOREM 5.** *Let  $D$  be an algebraic  $F$ -algebra and  $A = F[[x, y]]$ . If  $A \otimes_F D$  is local, then  $D$  is locally finite.*

**PROOF.** If  $0 \neq \alpha \in A$ , then  $\alpha = \sum_{i=0}^{\infty} z_i$ , where  $z_i$  is homogeneous of degree  $i$ . Let  $n(\alpha) = \min\{j: z_j \neq 0\}$ . Define  $v(\alpha) = 2^{-n(\alpha)}$ . Then  $v$  is a nonarchimedean valuation on  $A$ . Let  $\{d_i\}$  be an  $F$ -basis for  $D$ . Every element of  $A \otimes D$  can be written uniquely in the form  $\beta = \sum (a'_i \otimes d_i)$ . Define  $\text{supp } \beta = \{d_i: a'_i \neq 0\}$ . Now choose  $d_1, d_2, \dots, d_n \in \{d_i\}$  and let  $a_i = xx \cdots xy \cdots y$  be a monomial in  $A$  consisting of  $i$   $x$ 's followed by  $n+1-i$   $y$ 's. Clearly  $F[a_1, a_2, \dots, a_n]$  is a free  $F$ -algebra contained in  $A$ . If  $A \otimes D$  is local, then  $a_1 \otimes 1, \dots, a_n \otimes 1 \in J(A \otimes D)$ , hence

$$1 - \partial = 1 - \sum_{i=1}^n (a_i \otimes d_i)$$

is invertible in  $A \otimes D$ . Since  $F[a_1, \dots, a_n]$  is a free algebra, every element in  $\text{supp}(d_{i_1} d_{i_2} \cdots d_{i_n})$  is contained in  $\text{supp } \partial^n$ , where  $d_{i_j} \in \{d_1, \dots, d_n\}$ . By [1, Theorem 3.3],  $\bigcup_{n=1}^{\infty} \text{supp } \partial^n$  is finite. Therefore,  $d_1, \dots, d_n$  generate a finite dimensional subalgebra of  $D$ .

For the sake of completeness we include a proof that  $\bigcup_{n=1}^{\infty} \text{supp } \partial^n$  is finite. Using the unique representation of  $\beta$  in the previous theorem, define  $u(\beta) = \max\{v(a'_i)\}$ . If  $\beta$  and  $\beta'$  are elements of  $A \otimes D$ , then  $u(\beta + \beta') \leq \max\{u(\beta), u(\beta')\}$  and  $u(\beta'\beta) \leq u(\beta')u(\beta)$ . By hypothesis,  $1 - \partial$  is invertible with inverse  $\gamma$ . We claim that every element in the support of  $\partial^n$  is in the support of  $\gamma$ . If not suppose that  $d \in \text{supp } \partial^k$ ,  $d \notin \text{supp } \partial^m$  for all  $m < k$ , and  $d \notin \text{supp } \gamma$ . Let  $\zeta(k) = 1 + \partial + \cdots + \partial^k$ . Clearly  $u(\partial^{k+1}) \leq 2^{(n+1)(k+1)}$ , and  $u(\gamma - \zeta(k)) \geq 2^{(n+1)k}$ . We now have

$$\begin{aligned} 2^{(n+1)(k+1)} &\geq u(\partial^{k+1}) = u((1 - \partial)(\gamma - \zeta(k))) \\ &= u((\gamma - \zeta(k)) - \partial(\gamma - \zeta(k))) = u(\gamma - \zeta(k)), \end{aligned}$$

since

$$u(\gamma - \zeta(k)) > u(\partial)u(\gamma - \zeta(k)) \geq u(\partial(\gamma - \zeta(k))).$$

This contradiction completes the proof.

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