

## THE DIMENSION OF INVERSE LIMITS

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**ABSTRACT.** A result is obtained concerning the dimension of inverse limits of uniform spaces from which several propositions follow that improve earlier results by various authors.

**1. Introduction.** There are two obvious questions concerning inverse limits that arise in dimension theory: (a) What spaces  $X$  are inverse limits of "nice" spaces with dimension  $\leq$  dimension  $X$ ? (b) If  $X$  is the inverse limit of spaces with dimension  $\leq n$  (under what conditions) it is true that dimension  $X \leq n$ ? The first question has been dealt with by Mardešić [9], Pasynkov [15] and Kljušin [8], who generalised the result of Freudenthal asserting that a compact metrisable space  $X$  is the inverse limit of a sequence of polyhedra with  $\dim \leq \dim X$ . The second question has received less attention. We have the obvious result for compact spaces, and two results of Nagami, the first asserting that the inverse limit of a sequence of metric spaces of  $\dim \leq n$  has  $\dim \leq n$  [12], the second that if a countably paracompact space  $X$  is the inverse limit of a sequence of normal spaces with  $\dim \leq n$ , then, provided the bonding maps are open and surjective,  $X$  is normal and  $\dim X \leq n$  [13].

In this paper we show that if a uniform space  $(X, \mathcal{U})$  is the inverse limit of uniform spaces  $(X_\alpha, \mathcal{U}_\alpha)$  with  $\mathcal{U}_\alpha$ -dim  $X_\alpha \leq n$ , then  $\mathcal{U}$ -dim  $X \leq n$ , where  $\mathcal{U}$ -dim is the dimension function introduced by the author [2]. Some corollaries of this result are as follows. (1) The inverse limit of a sequence of perfectly normal spaces with  $\dim \leq n$  is perfectly normal with  $\dim \leq n$ . (2) If a strongly paracompact space  $X$  is the inverse limit of spaces with  $\dim \leq n$ , then  $\dim X \leq n$ . (3) If  $X$  is the product of separable metric spaces  $X_\alpha$  such that any finite product of  $X_\alpha$ 's has  $\dim \leq n$ , then  $X$  has dimension  $\leq n$ . For dimension zero, this result has also been obtained by Morita.

**2. Definitions.** A subset of a uniform space  $(X, \mathcal{U})$  is called  $\mathcal{U}$ -open ( $\mathcal{U}$ -closed) if it is the inverse image of an open (closed) subset of  $R$ , the space of real numbers, under a uniformly continuous function. In hyphenated words where " $\mathcal{U}_Z$ ", the relativisation of  $\mathcal{U}$  to a subset  $Z$  of  $X$ , occurs, we drop " $Z$ ". For example, " $\mathcal{U}_Z$ -open" becomes " $\mathcal{U}$ -open in  $Z$ ". A  $\mathcal{U}$ -cover (cozero cover) of  $X$  is a cover consisting of  $\mathcal{U}$ -open (cozero) sets of  $X$ . A dimension function  $d$  is simply a function with domain a class of spaces and range the set

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$\{-1, 0, 1, 2, \dots, \infty\}$  such that  $d(X) = -1$  if and only if  $X$  is empty. To define  $d$  it suffices to define " $d(X) \leq n$ " for  $n = 0, 1, \dots$  so that if  $d(X) \leq n$  and  $n \leq m$  then  $d(X) \leq m$ , for we can then let " $d(X) = n$ " mean " $d(X) \leq n$  and  $d(X) \not\leq n-1$ " and " $d(X) = \infty$ " mean " $d(X) \not\leq n, n = 0, 1, \dots$ ".

$\mathcal{U}$ -dim is defined on the class of all uniform spaces, and  $\dim, \dim^*$  on the class of all topological spaces as follows.

$\mathcal{U}$ -dim  $X \leq n$  ( $\dim X \leq n, \dim^* X \leq n$ ) if every finite  $\mathcal{U}$ -cover (open, cozero cover) of  $X$  is refinable by a finite  $\mathcal{U}$ -cover (open, cozero cover) of order  $\leq n$ . (A collection has order  $\leq n$  if any  $n+2$  of its elements have empty intersection.)

As is well known,  $\dim^*$  is the modification of  $\dim$  introduced by Katětov and Smirnov, and  $\dim = \dim^*$  for normal spaces. If  $\mathcal{U}$  is a metrisable uniformity on  $X$ , or the topology induced by  $\mathcal{U}$  on  $X$  is Lindelöf, or  $\mathcal{U}$  is the uniformity generated by the family of all (bounded) real-valued continuous functions on a topological space  $X$ , then the  $\mathcal{U}$ -open sets of  $X$  are precisely the cozero sets, and hence  $\mathcal{U}$ -dim  $X = \dim^* X$ . Furthermore, in the first two cases,  $X$  is normal and hence  $\mathcal{U}$ -dim  $X = \dim X$  [2].

Proofs of the following statements concerning  $\mathcal{U}$ -open sets of a uniform space  $(X, \mathcal{U})$ , which are used implicitly in the sequel, may be found in [2]. The collection of all  $\mathcal{U}$ -open ( $\mathcal{U}$ -closed) sets of  $X$  is closed with respect to finite intersections and countable unions (countable intersections and finite unions). If  $G$  is  $\mathcal{U}$ -open and  $F$  is  $\mathcal{U}$ -closed, then  $G - F$  is  $\mathcal{U}$ -open and  $F - G$  is  $\mathcal{U}$ -closed.  $H$  is  $\mathcal{U}$ -open ( $\mathcal{U}$ -closed) in a subspace  $Y$  of  $X$  if and only if  $H = G \cap Y$  for some  $\mathcal{U}$ -open ( $\mathcal{U}$ -closed) set  $G$  of  $X$ . If  $E, F$  are disjoint  $\mathcal{U}$ -closed sets of  $X$ , there are disjoint  $\mathcal{U}$ -open sets  $G, H$  of  $X$  with  $E \subset G$  and  $F \subset H$ . This will be referred to as the normality property of  $\mathcal{U}$ -open sets. Two subsets  $A, B$  of  $X$  are called distant if for some  $U$  in  $\mathcal{U}$ ,  $(A \times B) \cap U = \emptyset$ . Proofs of the following results can be found in [3].

LEMMA 1. *If  $A, B$  are distant sets of  $(X, \mathcal{U})$ , there are distant  $\mathcal{U}$ -closed sets  $E, F$  and distant  $\mathcal{U}$ -open sets  $G, H$  with  $A \subset E \subset G$  and  $B \subset F \subset H$ .*

LEMMA 2. *If  $A$  is a  $\mathcal{U}$ -closed set of  $X$ , there are  $\mathcal{U}$ -closed sets  $A_k, k = 1, 2, \dots$ , such that  $A = \bigcap A_k$  and  $A_{k+1}$  is distant from  $X - A_k$ .*

### 3. Dimension of inverse limits.

LEMMA 3. *Let  $(X, \mathcal{U})$  be the inverse limit of an inverse system of uniform spaces  $(X_\alpha, \mathcal{U}_\alpha, f_{\alpha\beta})$  with canonical projections  $f_\alpha: X \rightarrow X_\alpha$ , and suppose  $A, B$  are distant in  $X$ . Then for some  $\alpha, f_\alpha(A), f_\alpha(B)$  are distant in  $X_\alpha$ .*

PROOF. If  $g_\alpha = f_\alpha \times f_\alpha$ , by [1, Proposition 10, p. 181], for some  $\alpha$  and  $V$  in  $\mathcal{U}_\alpha$ ,  $(A \times B) \cap g_\alpha^{-1}(V) = \emptyset$ , and hence  $(f_\alpha(A) \times f_\alpha(B)) \cap V = \emptyset$ .

LEMMA 4. *Let  $(X, \mathcal{U}), (X, \mathcal{V})$  be the inverse limits of the inverse systems of uniform spaces  $(X_\alpha, \mathcal{U}_\alpha, f_{\alpha\beta})$  and  $(X_\alpha, \mathcal{V}_\alpha, f_{\alpha\beta})$ , respectively, and suppose that the  $\mathcal{U}_\alpha$ -open sets of each  $X_\alpha$  are precisely its  $\mathcal{V}_\alpha$ -open sets. Then the  $\mathcal{U}$ -open sets of  $X$  are precisely its  $\mathcal{V}$ -open sets.*

PROOF. For a given  $\mathcal{U}$ -open set  $G$  of  $X$ , let  $G_k, k = 1, 2, \dots$ , be  $\mathcal{U}$ -open sets such that  $G = \bigcup G_k$  and  $G_k$  is distant from  $X - G$  (Lemma 2). By Lemma 3,

for some  $\alpha = \alpha(k)$ ,  $f_\alpha(G_k)$ ,  $f_\alpha(X - G_k)$  are distant in  $X_\alpha$ . Hence there are disjoint  $\mathfrak{U}_\alpha$ -open sets  $P_k, Q_k$  of  $X_\alpha$  with  $f_\alpha(G_k) \subset P_k$  and  $f_\alpha(X - G_k) \subset Q_k$  (Lemma 1). By assumption  $P_k, Q_k$  are  $\mathfrak{V}_\alpha$ -open, and since  $f_\alpha: (X, \mathfrak{V}) \rightarrow (X_\alpha, \mathfrak{V}_\alpha)$  is uniformly continuous  $S_k = f_\alpha^{-1}(P_k)$ ,  $T_k = f_\alpha^{-1}(Q_k)$  are disjoint  $\mathfrak{V}$ -open sets of  $X$  with  $G_k \subset S_k$  and  $X - G_k \subset T_k$ . Hence  $G_k \subset S_k \subset G$ ,  $G = \bigcup S_k$  and  $G$  is  $\mathfrak{V}$ -open. The converse follows by symmetry.

In the sequel all spaces are invariably assumed to be Tychonoff.

**THEOREM.** *Let  $(X, \mathfrak{U})$  be the inverse limit of an inverse system of uniform spaces  $(X_\alpha, \mathfrak{U}_\alpha, f_{\alpha\beta})$  with  $\mathfrak{U}_\alpha - \dim X_\alpha \leq n$  for each  $\alpha$ . Then  $\mathfrak{U} - \dim X \leq n$ .*

**PROOF.** Let  $\mathfrak{V}_\alpha$  be the precompact uniformity on  $X_\alpha$  whose uniform covers are those that are refinable by the finite  $\mathfrak{U}_\alpha$ -covers of  $X_\alpha$ . Then the  $\mathfrak{V}_\alpha$ -open sets of  $X_\alpha$  are precisely its  $\mathfrak{U}_\alpha$ -open sets and

$$\delta d(X_\alpha, \mathfrak{V}_\alpha) = \mathfrak{V}_\alpha - \dim X_\alpha = \mathfrak{U}_\alpha - \dim X_\alpha \leq n$$

[2, Proposition 8], where  $\delta d$  denotes Smirnov's dimension function; see e.g. [7]. Clearly  $(X_\alpha, \mathfrak{V}_\alpha, f_{\alpha\beta})$  is an inverse system of uniform spaces, and if  $(X, \mathfrak{V})$  is its inverse limit, applying the inverse limit theorem for  $\delta d$  [7, p. 71], we obtain  $\delta d(X, \mathfrak{V}) \leq n$ . Finally, since  $\mathfrak{V} - \dim X \leq \delta d(X, \mathfrak{V})$  [2, Proposition 7], by Lemma 4,  $\mathfrak{U} - \dim X = \mathfrak{V} - \dim X \leq n$ .

Reference to the dimension function  $\delta d$  can be completely avoided by applying the obvious result for  $\dim$  on compact space to the inverse limit of the completions  $(Y_\alpha, \mathfrak{V}_\alpha)$  of  $(X_\alpha, \mathfrak{V}_\alpha)$  which satisfy  $\dim Y_\alpha = \mathfrak{V}_\alpha - \dim Y_\alpha = \mathfrak{U}_\alpha - \dim X_\alpha \leq n$  [2, Proposition 8], and then appealing to the subset theorem for  $\mathfrak{U} - \dim$  [2, Proposition 3]. Another similar proof can be obtained by using the interesting fact that  $\mathfrak{U} - \dim X = \delta d(X, m\mathfrak{U})$ , where  $(X, m\mathfrak{U})$  is the  $\mathfrak{U}$ -fine coreflection of  $(X, \mathfrak{U})$ ; see [21]. A uniform space is called  $\mathfrak{U}$ -fine, or metric-fine, if every uniformly continuous function into a metric uniform space remains uniformly continuous when the latter is re-equipped with its fine uniformity.  $m\mathfrak{U}$  is the coarsest  $\mathfrak{U}$ -fine uniformity which is finer than  $\mathfrak{U}$ . It can be shown that the precompact reflection of  $(X, m\mathfrak{U})$  has as a basis the collection of finite  $\mathfrak{U}$ -covers of  $X$ , which implies  $\delta d(X, m\mathfrak{U}) = \mathfrak{U} - \dim X$  and also that the  $m\mathfrak{U}$ -open sets of  $X$  are identical with the  $\mathfrak{U}$ -open sets. The same comments apply to the (separable metric)-fine coreflection of [20].

According to Smirnov [16], a Tychonoff space  $X$  has the monotonicity property relative to  $\dim^*$  if  $\dim^* X \leq \dim Y$  for every compactification  $Y$  of  $X$ . If  $\mathfrak{U}$  is a uniformity on  $X$ ,  $\mathfrak{V}$  is the precompact uniformity on  $X$  whose uniform covers are those that can be refined by finite  $\mathfrak{U}$ -covers, and  $(Y, \mathfrak{V})$  is the completion of  $(X, \mathfrak{V})$ , then  $\dim Y = \mathfrak{U} - \dim X$ . It follows that  $X$  has the monotonicity property relative to  $\dim^*$  if and only if  $\dim^* X \leq \mathfrak{U} - \dim X$  for every uniformity  $\mathfrak{U}$  on  $X$ . The class of all spaces with the monotonicity property relative to  $\dim^*$  contains all strongly paracompact spaces.

**PROPOSITION 1.** *Let  $X$  be the inverse limit of an inverse system of Tychonoff spaces  $X_\alpha$  with  $\dim^* X_\alpha \leq n$  for each  $\alpha$ , and suppose  $X$  has the monotonicity property relative to  $\dim^*$ . Then  $\dim^* X \leq n$ .*

**PROOF.** Let  $(X, \mathfrak{U})$  be the inverse limit of  $(X_\alpha, \mathfrak{U}_\alpha)$  where  $\mathfrak{U}_\alpha$  is the Čech

uniformity on  $X_\alpha$ , i.e. the uniformity generated by the family of all bounded real-valued continuous functions on  $X_\alpha$ . Then  $\mathfrak{U}_\alpha$ -dim  $X_\alpha = \dim^* X_\alpha \leq n$ , and by the theorem,  $\dim^* X \leq \mathfrak{U}$ -dim  $X \leq n$ .

**PROPOSITION 2.** *Let  $X$  be the inverse limit of an inverse system  $(X_n, f_{nm}, n \in N)$  of perfectly normal spaces with  $\dim X_n \leq k$  for each  $n$  in  $N$ , the set of natural numbers. Then  $X$  is perfectly normal and  $\dim X \leq k$ .*

**PROOF.** Let  $(X, \mathfrak{U})$  be the inverse limit of  $(X_n, \mathfrak{U}_n, f_{nm})$  where  $\mathfrak{U}_n$  is the Čech uniformity on  $X_n$ . Then  $\mathfrak{U}_n$ -dim  $X_n = \dim X_n \leq k$ . If  $A$  is closed in  $X$ , and  $f_n: X \rightarrow X_n$  are the canonical projections, then  $A = \bigcap_n f_n^{-1}(\overline{f_n(A)})$  [1, Corollary, p. 49]. Since  $X_n$  is perfectly normal,  $\overline{f_n(A)}$  is a zero set, and hence a  $\mathfrak{U}_n$ -closed set of  $X_n$ . Of course  $f_n$  is uniformly continuous, and hence  $f_n^{-1}(\overline{f_n(A)})$  is  $\mathfrak{U}$ -closed. Thus  $A$  is  $\mathfrak{U}$ -closed, and by the normality property of  $\mathfrak{U}$ -open sets,  $X$  is normal and hence perfectly normal. By our theorem,  $\dim X = \mathfrak{U}$ -dim  $X \leq k$ .

That the inverse limit of a sequence of perfectly normal spaces is perfectly normal has also been obtained by Cook and Fitzpatrick [4] by a relatively involved argument.

For the following results we need theorems giving sufficient conditions under which every real-valued continuous function on a subset  $Y$  of the product  $X$  of spaces  $X_\alpha$  depends on countably many coordinates, i.e. it is the composite of some real-valued continuous function on a countable subproduct  $Z$  of  $X_\alpha$ 's together with the canonical projection  $Y \rightarrow Z$ . In Gleason's theorem [7, p. 132], the conditions are that  $Y$  is an open set of  $X$  and each  $X_\alpha$  is separable. In Engelking's version of this theorem [5, Theorem 1], it is sufficient to assume that  $Y = X$  and that finite products of  $X_\alpha$ 's are Lindelöf. Noble and Ulmer [19] observe that for Engelking's proof to work it suffices to assume that finite products of  $X_\alpha$ 's have the property that each uncountable subset has an accumulation point, and this includes the case when finite products of  $X_\alpha$ 's are countably compact, which is used in the proof of Proposition 7.

**PROPOSITION 3.** *Let  $Y$  be an open subset of the product  $X$  of separable spaces  $X_\alpha$  such that any finite product of  $X_\alpha$ 's has  $\dim \leq n$ . Then  $\dim^* Y \leq n$  in each of the following two cases.*

- (1) *Any countable product of  $X_\alpha$ 's is hereditarily Lindelöf.*
- (2) *Any finite product of  $X_\alpha$ 's is perfectly normal.*

**PROOF.**  $X$  is the inverse limit of finite products  $Y_\beta$  of  $X_\alpha$ 's. If  $(X, \mathfrak{U})$  is the inverse limit of  $(Y_\beta, \mathfrak{U}_\beta)$  where  $\mathfrak{U}_\beta$  denotes the Čech uniformity on  $Y_\beta$ , then for each  $\beta$ ,  $\mathfrak{U}_\beta$ -dim  $Y_\beta = \dim Y_\beta \leq n$  and by our theorem  $\mathfrak{U}$ -dim  $X \leq n$ . By the subset theorem for  $\mathfrak{U}$ -dim [2],  $\mathfrak{U}$ -dim  $Y \leq n$ .

If  $f: Y \rightarrow R$  is continuous, by Gleason's theorem [7, p. 132] there is a countable product  $Z$  of  $X_\alpha$ 's and a continuous  $g: \pi(Y) \rightarrow R$  such that  $f = g\pi/Y$ , where  $\pi: X \rightarrow Z$  is the canonical projection. If  $Z$  is the inverse limit of spaces  $Y_\gamma$ , let  $\mathfrak{V}$  be the inverse limit of the corresponding uniformities  $\mathfrak{U}_\gamma$ . Then  $\pi: (X, \mathfrak{U}) \rightarrow (Z, \mathfrak{V})$  is uniformly continuous. Let  $H$  be an open set of  $R$ . In case (1), the cozero set  $g^{-1}(H)$  of the Lindelöf space  $\pi(Y)$  is  $\mathfrak{V}$ -open in  $\pi(Y)$ , and hence  $f^{-1}(H) = \pi^{-1}g^{-1}(H)$  is  $\mathfrak{U}$ -open in  $Y$ . In case (2),  $g^{-1}(H)$  is an

open set of  $Z$ , and since any open set of  $Z$  is  $\mathcal{U}$ -open (Proposition 2),  $f^{-1}(H)$  is  $\mathcal{U}$ -open in  $Y$ . It transpires that every cozero set of  $Y$  is  $\mathcal{U}$ -open, and hence  $\dim^* Y = \mathcal{U}\text{-dim } Y \leq n$ .

Using Engelking's [5] version of Gleason's theorem, we obtain

**PROPOSITION 4.** *Let  $X$  be the product of spaces  $X_\alpha$  every finite product of which has  $\dim \leq n$ . Then  $\dim^* X \leq n$  in each of the following two cases.*

- (1) *Countable products of  $X_\alpha$ 's are Lindelöf.*
- (2) *Finite products of  $X_\alpha$ 's are perfectly normal and Lindelöf.*

In case (1) of Proposition 4 for dimension 0, since the product theorem for  $\dim$  holds when the product is Lindelöf [12], we need only assume  $\dim X_\alpha \leq 0$  for each  $\alpha$ . In this form the result has also been obtained by Morita [11, Theorem 3]. The following result follows from Glicksberg's theorem [7, p. 138].

**PROPOSITION 5.** *If a pseudocompact space  $X$  is the product of spaces  $X_\alpha$ , every finite product of which has  $\dim^* \leq n$ , then  $\dim^* X \leq n$ .*

**PROPOSITION 6.** *Let a countably compact space  $X$  be the inverse limit of an inverse system of normal spaces and surjective maps  $(X_n, f_{nm})$  with  $\dim X_n \leq k$  for each  $n$ . Then  $X$  is normal and  $\dim X \leq k$ .*

**PROOF.** If  $E, F$  are disjoint closed sets of  $X$ ,  $f_n: X \rightarrow X_n$  the canonical projections, then  $\bigcap_n f_n^{-1}(f_n(E) \cap f_n(F)) = \emptyset$ , and since  $X$  is countably compact,  $f_n^{-1}(f_n(E) \cap f_n(F)) = \emptyset$  for some  $n$ . Since each  $f_{nm}$  is surjective, so is each  $f_n$  [7, p. 72] and hence  $f_n(E) \cap f_n(F) = \emptyset$ . It is now clear that since each  $X_n$  is normal, so is  $X$ , and if  $(X, \mathcal{U})$  is the limit of  $(X_n, \mathcal{U}_n)$  where  $\mathcal{U}_n$  denotes the Čech uniformity on  $X_n$ , then every open  $F_\sigma$ -set of  $X$  is  $\mathcal{U}$ -open. Our theorem then implies  $\dim X = \mathcal{U}\text{-dim } X \leq k$ .

**PROPOSITION 7.** *Let  $X$  be the product of spaces  $X_\alpha$  such that any countable product of  $X_\alpha$ 's is countably compact and any finite product of  $X_\alpha$ 's is normal with  $\dim \leq n$ . Then  $\dim^* X \leq n$ .*

**PROOF.** This follows from [19, Proposition 2.1], Proposition 6, and our theorem.

**4. Examples.** There are various directions in which our results may be conjectured to generalise, and the following examples rule out some of them. They also illustrate the usefulness of these results.

1. If  $X$  is the product of uncountably many copies of  $N$ , the space of natural numbers with the usual topology, Stone [17] has shown that  $X$  is not normal and hence  $\dim X > 0$ . Proposition 3 gives that any nonempty open set of  $X$  has  $\dim^* = 0$ . According to Morita, S. P. Franklin proved  $\dim^* X = 0$  in 1970.

2. If  $R_\omega$  is the set of points of the Hilbert space all of whose coordinates are rational, it is well known that  $\dim R_\omega = 1$  and  $R_\omega \times R_\omega = R_\omega$  [6]. It follows from Proposition 3 that any nonempty open subset of the product of any number of copies of  $R_\omega$  has  $\dim^* = 1$ .

3. For each ordinal  $\alpha < \omega_1$ , the first uncountable ordinal, let  $I_\alpha$  be a subset of  $I^n$ ,  $n$  a fixed element of  $\{1, 2, \dots, \infty\}$ , such that  $0 \in I_\alpha$ ,  $\dim I_\alpha = 0$ , for

$\alpha \leq \beta$ ,  $I_\alpha \subset I_\beta$ , and  $I^n = \bigcup_{\alpha < \omega_1} I_\alpha$  [12, Theorems 13–15]. Let  $M_\alpha$ ,  $M$  be the following subspaces of  $[0, \omega_1] \times I^n$ .

$$M_\alpha = \bigcup_{\gamma \leq \alpha} \{\gamma\} \times I_\gamma \cup \{(\beta, 0) : \beta < \omega_1\},$$

$$M = \bigcup_{\alpha < \omega_1} M_\alpha = \bigcup_{\alpha < \omega_1} \{\alpha\} \times I_\alpha.$$

Define continuous  $f_{\alpha\beta}: M_\beta \rightarrow M_\alpha$ ,  $\alpha \leq \beta$ , and  $f_\alpha: M \rightarrow M_\alpha$  by  $f_{\alpha\beta}(\gamma, x) = (\gamma, x)$  if  $\gamma \leq \alpha$ ,  $f_{\alpha\beta}(\gamma, x) = (\gamma, 0)$  if  $\gamma > \alpha$ ,  $f_\alpha(\gamma, x) = (\gamma, x)$  if  $\gamma \leq \alpha$ , and  $f_\alpha(\gamma, x) = (\gamma, 0)$  if  $\gamma > \alpha$ . Then  $M$  is the inverse limit of the inverse system  $(M_\alpha, f_{\alpha\beta})$  with canonical projections  $f_\alpha$ , and  $f_\alpha, f_{\alpha\beta}$  are both open and surjective. Yet  $M$  is a normal space with  $\dim M = n$  [12] while each  $M_\alpha$  is completely normal with  $\dim M_\alpha = 0$ .

4. If  $X$  is the inverse limit of an inverse system of normal spaces and open surjective maps  $(X_n, f_{nm})$  with  $\dim X_n \leq k$  for each  $N$ , is it true that  $\dim^* X \leq k$ ? Nagami's result [13, Theorem 17] asserts that this is so provided that  $X$  is countably paracompact. Let  $X$  be Michael's hereditary paracompact and Lindelöf space with  $\dim = 0$  [10], and  $Y$  be the irrationals. Then, as Nagami [13] observes,  $Y$  is the inverse limit of countable discrete spaces  $Y_i$ ,  $i = 1, 2, \dots$ , and hence  $X \times Y$  is the inverse limit of hereditary paracompact and Lindelöf spaces  $X \times Y_i$  with  $\dim(X \times Y_i) = 0$  (and open surjective bonding maps). Yet  $X \times Y$  is not normal, and hence  $\dim X \times Y > 0$ . An affirmative answer to our question implies  $\dim^* X \times Y = 0$ , and this has been shown to be true [18]. An affirmative answer in fact implies: If  $X$  is a normal space and  $Y$  is a zero-dimensional separable and completely metrisable space, then  $\dim^* X \times Y = \dim X$ .

5. Can we extend Proposition 4 to arbitrary inverse limits of metric separable spaces? Mrowka is reported to have obtained an example of a closed subspace  $X$  of an infinite product of copies of  $N$  with  $\dim^* X > 0$  [14, §3]. Then  $X$  is the inverse limit of (zero-dimensional) countable discrete spaces, and yet  $\dim^* X > 0$ .

6. If a completely normal first countable space  $X$  is the inverse limit of completely normal first countable spaces  $X_\alpha$  with  $\dim X_\alpha \leq n$ , is it true that  $\dim X \leq n$ ? Example 3 shows that this is false if we drop the assumption that  $X$  is completely normal. Let  $A$  be a subset of a completely normal first countable space  $X$ , and for each  $x$  in  $X - A$ , let  $A_x = X - \{x\}$ . Then the normal space  $A_x$  is the countable union of closed sets with  $\dim \leq \dim X$  and hence  $\dim A_x \leq \dim X$ . Since  $A$  is the inverse limit of the cofinite subsets of  $X$  containing  $A$ , if the answer to our question is affirmative, then the subset theorem for completely normal first countable spaces is valid. The latter is an open question in dimension theory.

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