PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

ON THE REALIZATION OF THE METRIC-DEPENDENT DIMENSION FUNCTION d_2

DEDICATED TO KIITI MORITA ON HIS 60TH BIRTHDAY

TATSUO GOTO

ABSTRACT. Let (X, ρ) be a metric space with $d_2(X,\rho) < \dim X$ where d_2 denotes the metric-dependent dimension function introduced by K. Nagami and J. H. Roberts [2]. Then it will be shown that for any integer k with $d_2(X,\rho) \leqslant k \leqslant \dim X$ there exists a topologically equivalent metric ρ_k with $d_2(X,\rho_k) = k$. This extends a result of J. C. Nichols [3] and answers the problem raised by K. Nagami and J. H. Roberts [2] in the affirmative.

1. **Introduction.** Let (X, ρ) be a nonempty metric space. Then the metric-dependent dimension $d_2(X,\rho)$ is defined to be at most n if for any collection $\{(C_i, C'_i): i = 1, \ldots, n+1\}$ of $n+1$ pairs of closed sets with $\rho(C_i, C'_i) > 0$, there exist closed sets B_i, $i = 1, \ldots, n+1$, such that (i) B_i separates X between C_i and C'_i for each i, and (ii) $\bigcap_{i=1}^{n+1} B_i = \varnothing$, where $\rho(C_i, C'_i)$ denotes the distance between C_i and C'_i w.r.t. the metric ρ (cf. [2]). In this paper the following realization theorem for d_2 will be proved, which answers the problem posed by [2] in the affirmative.

THEOREM. *Let (X, ρ) be a metric space with $d_2(X,\rho) < \dim X$, and let k be any integer with $d_2(X,\rho) \leqslant k \leqslant \dim X$. Then there exists a topologically equivalent metric ρ_k such that $d_2(X,\rho_k) = k$.*

A similar assertion was proved by J. C. Nichols [3] in the case where (X, ρ) belongs to some special class of spaces which are subsets of finite dimensional Euclidean cubes with the usual metrics.

Throughout the paper, (X, ρ) denotes a metric space with a metric ρ which is compatible with the topology of X.

2. **Some lemmas.** Let (C, C') be a pair of closed sets of (X, ρ) with $\rho(C, C') > 0$; then there exists a uniformly continuous mapping f of (X, ρ) into the closed unit interval I such that $f(C) = 0$ and $f(C') = 1$. More generally we shall prove

LEMMA 1. *Let (C, C') be a pair of disjoint closed sets and $\{W_i: i = 1, 2, \ldots\}$ a decreasing sequence of open sets in (X, ρ) such that*

(i) $\rho(W_{i+1}, X \setminus W_i) > 0$ *for all* i,

(ii) $\bigcap_{i=1}^{\infty} W_i = \varnothing$,

(iii) $\rho(C \setminus W_i, C' \setminus W_i) > 0$ *for all* i.

Received by the editors September 19, 1975.
AMS (MOS) subject classifications (1970). Primary 54F45, 54E35; Secondary 54C30.
Key words and phrases. Metric spaces, metric-dependent dimension d_2, realizations of d_2.

Then there exists a continuous mapping $f\colon X \to I$ *such that* $f(C) = 0, f(C') = 1$ *and* $f|(X \setminus W_i)$ *is uniformly continuous w.r.t.* ρ *for each* i.

PROOF. By assumption (i) we can choose open sets U_i and V_i, $i = 1, 2, \ldots$, such that

$$(1) \qquad W_i \subset U_i \subset V_i \subset W_{i-1}, \quad W_0 = X,$$

with $\rho(W_i, X \setminus U_i) > 0$, $\rho(U_i, X \setminus V_i) > 0$ and $\rho(V_i, X \setminus W_{i-1}) > 0$ for each i.

Then for each i there exists a uniformly continuous mapping $\tau_i\colon (X, \rho) \to I$ such that

$$(2) \qquad \tau_i(U_i) = 0 \quad \text{and} \quad \tau_i(X \setminus V_i) = 1.$$

Similarly by condition (iii) there exists a uniformly continuous mapping $g_i\colon (X \setminus W_i, \rho) \to I$ for each i such that

$$(3) \qquad g_i(C \setminus W_i) = 0 \quad \text{and} \quad g_i(C' \setminus W_i) = 1.$$

Let us define $f_i\colon W_{i-1} \setminus W_i \to I$ by

$$(4) \qquad f_i(x) = \tau_i(x) g_i(x) + (1 - \tau_i(x)) g_{i+1}(x), \qquad x \in W_{i-1} \setminus W_i.$$

Then clearly f_i is uniformly continuous w.r.t. ρ by the uniform continuity of τ_i, g_i and g_{i+1}. From (2) it follows that

$$(5) \qquad f_i|(W_{i-1} \setminus V_i) = g_i \quad \text{and} \quad f_i|(U_i \setminus W_i) = g_{i+1} \quad \text{for all } i.$$

Moreover by (3) and (4) we obtain

$$(6) \qquad f_i(C \cap (W_{i-1} \setminus W_i)) = 0 \quad \text{and} \quad f_i(C' \cap (W_{i-1} \setminus W_i)) = 1.$$

Since $X = \bigcup_{i=1}^{\infty} (W_{i-1} \setminus W_i)$ by assumption (ii), we can define a mapping $f\colon X \to I$ by

$$(7) \qquad f|(W_{i-1} \setminus W_i) = f_i \quad \text{for } i = 1, 2, \ldots.$$

Then the continuity of f is easily proved by use of (1) and (5). Furthermore, it follows from (6) that $f(C) = 0$ and $f(C') = 1$. To prove that

$$(8) \qquad f|(X \setminus W_i)\colon (X \setminus W_i, \rho) \to I \quad \text{is uniformly continuous for all } i,$$

consider a covering $\mathfrak{U} = \{W_{j-1} \setminus W_j : j = 1, \ldots, i\} \cup \{U_j \setminus V_{j+1} : j = 1, \ldots, i - 1\}$ of $X \setminus W_i$. Then obviously (1) implies that $\mathfrak{U}$ is a uniform covering of $(X \setminus W_i, \rho)$. On the other hand, from (5) we can deduce that $f|(U_j \setminus V_{j+1}) = g_{j+1}$ for each j, which combined with (7) implies that $f|U$ is uniformly continuous w.r.t. ρ for each $U \in \mathfrak{U}$. Since $\mathfrak{U}$ is a uniform covering, we obtain (8) and the proof of the lemma is completed.

DEFINITION 1. Let ρ and ρ' be metrics on X compatible with the topology of X such that the identity mapping of (X, ρ) onto (X, ρ') is uniformly continuous; then we denote this relation by $\rho \succ \rho'$.

By the definition it is clear that $d_2(X, \rho) \geqslant d_2(X, \rho')$ if $\rho \succ \rho'$.

LEMMA 2. *Let* $\{(C_i, C_i') : i = 1, 2, \ldots, n\}$ *be a collection of* n *pairs of disjoint* *closed sets of* (X, ρ). *Then there exists a metric* σ *on* X *and continuous mappings* $f_i: X \to I$, $i = 1, \dots, n$, *such that*

(i) σ *is compatible with the topology of* X *and* $\rho \succ \sigma$,

(ii) $f_i(C_i) = 0$ *and* $f_i(C'_i) = 1$,

(iii) *for any* $\varepsilon > 0$ *there exists an open set* U *of* X *such that* $\sigma(U) \leqslant \varepsilon$ *and* $f_i|(X \setminus U)$: $(X \setminus U, \sigma) \to I$ *is uniformly continuous for all* i.

Here $\sigma(U)$ denotes the diameter of U w.r.t. σ.

PROOF. We define $A_{k,i} = \{x \in X: \rho(x, C_i) \leqslant 1/k, \rho(x, C'_i) \leqslant 1/k\}$ and $A_k = \bigcup_{i=1}^n A_{k,i}$. Then we have

$$A_{k+1} \subset A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k = \varnothing. \tag{1}$$

To show the latter half, let x be an arbitrary point of X. Then for each i there exists $k(i)$ such that $x \notin A_{k(i),i}$ by the assumption that $C_i \cap C'_i = \varnothing$. Hence if we put $k = \max\{k(i): i = 1, \dots, n\}$, then we have $x \notin \bigcup_{i=1}^n A_{k,i} = A_k$ since $A_{k,i} \subset A_{k(i),i}$ for all i. Thus we obtain $\bigcap_{k=1}^{\infty} A_k = \varnothing$.

Suppose that $A_k = \varnothing$ for some k. Then we have $A_{k,i} = \varnothing$ and hence $\rho(C_i, C'_i) \geqslant 1/k$ for all i. Therefore there exist uniformly continuous mappings $f_i: (X, \rho) \to I$ such that $f_i(C_i) = 0$ and $f_i(C'_i) = 1$, which obviously satisfy condition (iii) of the lemma. Hence we may assume that $A_k \neq \varnothing$ for all k.

Now let $\{\mathcal{U}_i: i = 1, 2, \dots\}$ be a sequence of uniform open coverings of (X, ρ) such that

$$\mathcal{U}_{i+1} \overset{*}{<} \mathcal{U}_i, \text{ that is, } \mathcal{U}_{i+1} \text{ is a star-refinement of } \mathcal{U}_i, \text{ for each } i, \tag{2}$$

$$\rho\text{ - mesh } \mathcal{U}_i < 1/i, \qquad i, 1, 2, \dots. \tag{3}$$

We let

$$B_k = S(A_k, \mathcal{U}_k) = \bigcup\{U \in \mathcal{U}_k: U \cap A_k \neq \varnothing\}$$

and

$$\mathcal{V}_k = \{B_k\} \cup \{U \in \mathcal{U}_k: U \cap A_k = \varnothing\} \quad \text{for } k = 1, 2, \dots.$$

Clearly we have

(4) $\mathcal{V}_k$ is a uniform covering of (X, ρ) for all k.

Moreover we shall prove

$$\mathcal{V}_{k+1} \overset{*}{<} \mathcal{V}_k \quad \text{for each } k, \tag{5}$$

$$\text{for each point } x \text{ of } X, \{S(x, \mathcal{V}_k): k = 1, 2, \dots\} \text{ forms a neighborhood basis at } x. \tag{6}$$

To prove (5), let $V \in \mathcal{V}_{k+1}$. In case $V = B_{k+1}$,

$$S(V, \mathcal{V}_{k+1}) = S(S(A_{k+1}, \mathcal{U}_{k+1}), \mathcal{U}_{k+1}) \subset S(A_k, \mathcal{U}_k) = B_k \in \mathcal{V}_k$$

by (1) and (2). If $V \neq B_{k+1}$, then $V \in \mathfrak{U}_{k+1}$ with $V \cap A_{k+1} = \varnothing$. Hence $S(V, \mathfrak{V}_{k+1}) = S(V, \mathfrak{U}_{k+1})$ if $S(V, \mathfrak{V}_{k+1}) \cap A_{k+1} = \varnothing$, and

$$S(V, \mathfrak{V}_{k+1}) \subset S(A_{k+1}, \mathfrak{U}^*_{k+1}) \subset S(A_k, \mathfrak{U}_k) = B_k$$

if $S(V, \mathfrak{V}_{k+1}) \cap A_{k+1} \neq \varnothing$. Thus in any case $S(V, \mathfrak{V}_{k+1})$ is contained in some element of $\mathfrak{V}_k$, and hence (5) is proved.

To show (6), let N be an arbitrary neighborhood of x. Then from (1) it follows that $x \in X \backslash A_i$ for some i. Since A_i is closed, we have $S(x, \mathfrak{U}_k) \subset N \cap (X \backslash A_i)$ for some $k \geqslant i$ by (3), which implies $S(x, \mathfrak{V}_k) = S(x, \mathfrak{U}_k) \subset N$. Thus (6) is proved.

In view of (5) and (6), it can be shown that there exists a metric σ on X compatible with the topology of X satisfying the condition

$$\text{(7)} \qquad \mathfrak{V}_{i+1} < \{S_{2^{-i}}(x, \sigma) \colon x \in X\} < \mathfrak{V}_i \quad \text{for } i = 1, 2, \ldots,$$

where $S_\varepsilon(x, \sigma)$ denotes the spherical neighborhood $\{x' \in X \colon \sigma(x, x') < \varepsilon\}$ (cf. [1, Theorem 4]). It is clear that (7) combined with (4) implies condition (i) of the lemma. Moreover we shall prove

$$\text{(8)} \qquad S(C_i \backslash B_k, \mathfrak{V}_k) \cap (C'_i \backslash B_k) = \varnothing \quad \text{for each } i \text{ and } k.$$

Let us assume the contrary. Then there exists $V \in \mathfrak{V}_k$ with $V \neq B_k$ such that $V \cap (C_i \backslash B_k) \neq \varnothing$ and $V \cap (C'_i \backslash B_k) \neq \varnothing$. But then $\rho(C_i \backslash B_k, C'_i \backslash B_k) < 1/k$ by (3), which contradicts the fact that $\rho(C_i \backslash B_k, C'_i \backslash B_k) \geqslant \rho(C_i \backslash A_k, C'_i \backslash A_k) \geqslant 1/k$. Hence (8) is proved.

From (7) and (8) it follows that $\sigma(C_i \backslash B_k, C'_i \backslash B_k) \geqslant 2^{-k}$.

Now we let $W_k = S_{2^{2-k}}(A_k, \sigma)$ for $k = 1, 2, \ldots$. Then we have $B_k \subset W_k$ because $\sigma(B_k) \leqslant 2^{2-k}$ by (7), and hence we obtain

$$\text{(9)} \qquad \sigma(C_i \backslash W_k, C'_i \backslash W_k) \geqslant 2^{-k} \quad \text{for each } i \text{ and } k.$$

Furthermore we assert that

$$\text{(10)} \qquad \sigma(W_{k+1}, X \backslash W_k) > 0 \quad \text{and} \quad \bigcap_{k=1}^{\infty} W_k = \varnothing.$$

For, since $A_{k+1} \subset A_k$, we have

$$\begin{aligned} \sigma(W_{k+1}, X \backslash W_k) &= \sigma(S_{2^{1-k}}(A_{k+1}, \sigma), X \backslash W_k) \geqslant \sigma(S_{2^{1-k}}(A_k, \sigma), X \backslash W_k) \\ &\geqslant 2^{2-k} - 2^{1-k} > 0. \end{aligned}$$

To prove the latter half of (10), assume the contrary. Then there exists a point $x_0 \in \bigcap_{k=1}^{\infty} W_k$. Since $\lim_{k \to \infty} \sigma(A_k, X \backslash W_k) = 0$ and $\{A_k\}$ is decreasing, we have $x_0 \in \bigcap_{k=1}^{\infty} A_k$. But this contradicts (1), and hence (10) is proved.

By virtue of (9) and (10), we can apply Lemma 1 for each pair (C_i, C'_i) and we get continuous mappings $f_i \colon X \to I$ for $i = 1, \ldots, n$ such that $f_i(C_i) = 0$, $f_i(C'_i) = 1$ and $f_i | (X \backslash W_k)$ is uniformly continuous w.r.t. σ for each i and k. Since $\lim_{k \to \infty} \sigma(W_k) = 0$, each f_i satisfies conditions (ii) and (iii) of the lemma, and hence the proof of the lemma is completed.

Let τ be the metric on the k-dimensional cube I^k such that $\tau(x, x')$

$= \sum_{i=1}^{k} |x_i - x'_i|$ where $x = (x_i)$, $x' = (x'_i)$.

DEFINITION 2. Let f be a continuous mapping of (X, ρ) into I^k. Then $\rho(f)$ denotes a metric on X defined by $\rho(f)(x, x') = \rho(x, x') + \tau(f(x), f(x'))$ for $x, x' \in X$.

Clearly $\rho(f)$ is compatible with the topology of X, and it is easy to see that $d_2(X, \rho(f)) \geqslant d_2(X, \rho)$ since $\rho(f) \succ \rho$. The following lemma is an extension of [3, Theorem 3].

LEMMA 3. *Let (X, σ) be a metric space and f a continuous mapping of X into I^k satisfying the condition that for any $\varepsilon > 0$ there exists an open set U of X such that*

(i) $\sigma(U) \leqslant \varepsilon$.

(ii) $f|(X \setminus U)$: $(X \setminus U, \sigma) \to I^k$ *is uniformly continuous. Then* $d_2(X, \sigma(f)) \leqslant \max\{d_2(X, \sigma), k\}$.

PROOF. Let us put $n = \max\{d_2(X, \sigma), k\}$, and let $\{(C_i, C'_i): i = 1, \ldots, n+1\}$ be an arbitrary collection of $n + 1$ pairs of closed sets with $\sigma(f)(C_i, C'_i) \geqslant \varepsilon > 0$ for all i. Then by the assumption, there exists an open set U of X such that

(1) $\sigma(U) \leqslant \varepsilon/4$ and $f|(X \setminus U)$ is uniformly continuous w.r.t. σ.

Then $\sigma(f)$ is uniformly equivalent to σ on $X \setminus U$, and hence we have

$$\sigma(C_i \setminus U, C'_i \setminus U) > 0 \quad \text{for all } i. \tag{2}$$

We let $B = S_{\varepsilon/4}(U, \sigma)$; then we know that

$$\tau(f(C_i \cap V), f(C'_i \cap V)) \geqslant \varepsilon/4 \quad \text{for each } i \tag{3}$$

where τ is the metric on I^k as in Definition 2. To prove (3), let $x \in C_i \cap V$ and $x' \in C'_i \cap V$. Then since $\sigma(f)(C_i, C'_i) \geqslant \varepsilon$, we get

$$\sigma(f)(x, x') = \sigma(x, x') + \tau(f(x), f(x')) \geqslant \varepsilon.$$

But $\sigma(x, x') \leqslant \sigma(V) \leqslant \sigma(U) + \varepsilon/2 \leqslant 3\varepsilon/4$ by (1), and hence $\tau(f(x), f(x')) \geqslant \varepsilon/4$, which means (3).

Since $n \geqslant k$, we can deduce from (3) that there exist closed sets $\tilde{D}_i$ of I^k, $i = 1, \ldots, n+1$, such that

(4) $I^k \setminus \tilde{D}_i$ can be expressed as a union of disjoint open sets $\tilde{E}_i$ and $\tilde{E}'_i$ with $f(C_i \cap V) \subset \tilde{E}_i$ and $f(C'_i \cap V) \subset \tilde{E}'_i$,

$$\bigcap_{i=1}^{n+1} \tilde{E}_i = \varnothing. \tag{5}$$

Then we can take open sets $\tilde{M}_i$ in I^k, $i = 1, \ldots, n+1$, such that

(6)
$$\bigcap_{i=1}^{n+1} \mathrm{Cl}(\tilde{M}_i) = \varnothing \quad \text{and}$$
$$\tilde{D}_i \subset \tilde{M}_i \subset \mathrm{Cl}(\tilde{M}_i) \subset I^k \setminus \{f(C_i \cap V) \cup f(C'_i \cap V)\} \quad \text{for each } i.$$

We let $\tilde{N}_i = \tilde{E}_i \backslash \tilde{M}_i$ and $\tilde{N}'_i = \tilde{E}'_i \backslash \tilde{M}_i$ for all i. Then $\tilde{N}_i$ and $\tilde{N}'_i$ are disjoint closed sets in I^k. Hence we have $\tau(\tilde{N}_i, \tilde{N}'_i) > 0$, which implies that

$$\sigma(N_i \backslash U, N'_i \backslash U) > 0 \quad \text{for all } i, \tag{7}$$

where $N_i = f^{-1}(\tilde{N}_i)$ and $N'_i = f^{-1}(\tilde{N}'_i)$, because

$$\sigma(f)(N_i \backslash U, N'_i \backslash U) \geqslant \tau(\tilde{N}_i, \tilde{N}'_i)$$

and σ is uniformly equivalent to $\sigma(f)$ on $X \backslash U$.

Since $\sigma(U, X \backslash V) \geqslant \varepsilon/4$, we can choose a closed set W in X such that $\sigma(U, X \backslash W) > 0$ and $\sigma(W, X \backslash V) > 0$. Now we define

$$P_i = ((N_i \cap W) \cup C_i) \backslash U \quad \text{and} \quad P'_i = ((N'_i \cap W) \cup C'_i) \backslash U$$

for $i = 1, \ldots, n+1$. Then P_i and P'_i are disjoint closed sets of $X \backslash U$ and

$$\sigma(P_i, P'_i) \geqslant \min\{\sigma(N_i \backslash U, N'_i \backslash U), \sigma(W, X \backslash V), \sigma(C_i \backslash U, C'_i \backslash U)\} > 0$$

by (2) and (7). Since $d_2(X \backslash U, \sigma) \leqslant d_2(X, \sigma) \leqslant n$, we can choose closed sets A_i in $X \backslash U$, $i = 1, \ldots, n+1$, such that

$$A_i \text{ separates } X \backslash U \text{ between } P_i \text{ and } P'_i \text{ for all } i, \tag{8}$$

$$\bigcap_{i=1}^{n+1} A_i = \varnothing. \tag{9}$$

Finally we define $B_i = f^{-1}(\mathrm{Cl}\,(\tilde{M}_i)) \cap W$ for $i = 1, \ldots, n+1$. Then in view of (4) and (6) we know that

$$B_i \text{ separates } W \text{ between } C_i \cap W \text{ and } C'_i \cap W \text{ for all } i, \tag{10}$$

$$\bigcap_{i=1}^{n+1} B_i = \varnothing. \tag{11}$$

Since $A_i \cap W \subset B_i$ for each i, it follows from (8) and (10) that

$$A_i \cup B_i \text{ separates } X \text{ between } C_i \text{ and } C_i \text{ for all } i. \tag{12}$$

It remains to prove that

$$\bigcap_{i=1}^{n+1} (A_i \cup B_i) = \varnothing. \tag{13}$$

Since $(A_i \cup B_i) \cap W = B_i$, we have $\bigcap_{i=1}^{n+1} (A_i \cup B_i) \cap W = \varnothing$ by (11). On the other hand $\bigcap_{i=1}^{n+1} (A_i \cup B_i) \backslash W \subset \bigcap_{i=1}^{n+1} A_i = \varnothing$ by (9), which proves (13). Therefore we can conclude that $d_2(X, \sigma(f)) \leqslant n$, and the proof of the lemma is completed.

3. **Main theorem.** Now we shall proceed to the proof of the following theorem.

THEOREM. *Let (X, ρ) be a metric space with $d_2(X, \rho) <$ dim X and let k be any integer with $d_2(X, \rho) \leqslant k \leqslant$ dim X. Then there exists a topologically equivalent metric ρ_k such that $d_2(X, \rho_k) = k$.*

PROOF. Let us put $n = \dim X$. Then there exists a collection $\{(C_i, C'_i): i = 1, \ldots, n\}$ of n pairs of disjoint closed sets of X such that

(1) if B_i separates X between C_i and C'_i for each i, then $\bigcap_{i=1}^{n} B_i \neq \varnothing$.

Then by Lemma 2 there exists a metric σ on X and continuous mappings $f_i: X \to I$, $i = 1, \ldots, n$, such that

(2) σ is compatible with the topology of X and $\rho \succ \sigma$,

(3) $$f_i(C_i) = 0 \quad \text{and} \quad f_i(C'_i) = 1,$$

(4) for any $\varepsilon > 0$ there exists an open set U of X such that $\sigma(U) \leqslant \varepsilon$ and $f_i|(X \setminus U)$ is uniformly continuous w.r.t. σ for all i.

For each k with $d_2(X, \rho) \leqslant k \leqslant n$, let us define $g_k: X \to I^k$ by $g_k(x) = (f_1(x), \ldots, f_k(x))$. Then by (4), g_k satisfies the condition that

(5) for any $\varepsilon > 0$ there exists an open set U of X such that $\sigma(U) \leqslant \varepsilon$ and $g_k|(X \setminus U)$ is uniformly continuous w.r.t. σ.

Hence we can apply Lemma 3 to g_k, and we obtain $d_2(X, \sigma(g_k)) \leqslant \max\{d_2(X, \sigma), k\}$. Since $\rho \succ \sigma$ by (2), we have $d_2(X, \rho) \geqslant d_2(X, \sigma)$, which implies that

(6) $$d_2(X, \sigma(g_k)) \leqslant k.$$

In view of (3) we know that $\sigma(g_k)(C_i, C'_i) \geqslant 1$ for $i = 1, \ldots, k$. Then we deduce from (1) that $d_2(X, \sigma(g_k)) \geqslant k$. Combining with (6), we conclude that $d_2(X, \rho_k) = k$ where $\rho_k = \sigma(g_k)$, which completes the proof of the theorem.

REFERENCES

1. J. R. Isbell, *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R.I., 1964. MR **30** #561.

2. K. Nagami and J. H. Roberts, *A study of metric-dependent dimension functions*, Trans. Amer. Math. Soc. **129** (1967), 414–435. MR **35** #6131.

3. J. C. Nichols, *The realization of dimension function d_2*, Fund. Math. **77** (1973), 211–217. MR **48** #3023.

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, SAITAMA UNIVERSITY, URAWA, JAPAN