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AN A-PROPER MAP WITH PRESCRIBED TOPOLOGICAL DEGREE

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ABSTRACT. For any given element α of the ring $^*\mathbf{Z} = \mathbf{Z}^{\mathbf{N}}/I$, where I is the ideal of integer sequences convergent to 0, an A-proper map in l_2 is constructed whose degree in the sense of S. F. Wong is equal to α.

The concept of an A-proper map acting in a Banach space was introduced by Petryshyn [6, p. 157] (originally called an operator satisfying condition (H)). A topological degree for such maps was defined by Browder and Petryshyn [1], [2] as a generalization of the Leray-Schauder degree [5] for maps of the form Identity + Compact. Browder and Petryshyn established the basic properties of their degree, Deg (T, G, y), which is set valued (see the note after Definition 3 for its definition), invariant under suitable homotopies, satisfies a weakened sum formula

$$\operatorname{Deg}(T, G, y) \subseteq \operatorname{Deg}(T, G_1, y) + \operatorname{Deg}(T, G_2, y),$$

and if Deg $(T, G, y) \neq \{0\}$ then there is an $x \in G$ such that $T(x) = y$. Wong [9] has given a new definition of the degree with values in a ring $^*\mathbf{Z}$ (see below) which satisfies the sum formula with an equality sign. Later Wong [10] proved a restricted product formula for the degree of the product TU under the restriction that at least one of the maps T or U must be of the form Identity + Compact.

The purpose of this paper is, given $\alpha \in {}^*\mathbf{Z}$, to construct an A-proper map in l_2 whose degree is α. In the following let X be a real Banach space. Let cl (G) denote the closure of G and ∂G the topological boundary of G for subsets G of X.

DEFINITION 1. An (oriented) projectionally complete scheme Γ for mappings from subsets of X to X is a monotonically increasing sequence $\{X_n\}$ of (oriented) finite dimensional subspaces of X and a sequence $\{P_n\}$ of continuous linear projections $P_n: X \to X_n$ with $P_n X = X_n$, such that $P_n x \to x$ as $n \to \infty$ for each $x \in X$.

This definition is adapted from Fitzpatrick [4, Definition 1.1, p. 537]. The following definition is that of Petryshyn [7, Definition 1, p. 271].

DEFINITION 2. Let G be a subset of X, and Γ a projectionally complete

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scheme in the sense of Definition 1. The mapping T: cl $(G) \to X$ is A-proper with respect to Γ if for any bounded sequence $\{x_{n_j}\}$ with $x_{n_j} \in \text{cl}\,(G) \cap X_{n_j}$ such that $P_{n_j} T(x_{n_j}) \to g \in X$, there exists a subsequence $\{x_{n_{j(k)}}\}$ and an $x \in \text{cl}\,(G)$ such that $x_{n_{j(k)}} \to x$ as $k \to \infty$ and $T(x) = g$.

Such mappings include mappings of the form $I + C$ where I is the identity and C is compact [6, Remark 3, p. 162], mappings of the form $I + S + C$ under certain conditions where S is strictly contractive [6, Theorem 7, p. 162], and K-monotone mappings under certain conditions [7, Corollary 2.1, p. 220 and Theorem 2.3, p. 222]. This latter class includes monotone mappings [7, p. 228] and J-monotone or accretive mappings [7, pp. 230–231].

The following definition is adapted from Wong [9, p. 406] and makes use of the classical degree in R^n, deg (f, D, q), called the Brouwer degree of f at q relative to D (cf. [3, Definition 6.3, p. 31] or [8, Definition 3.14, p. 71]). Here D is a bounded open set in oriented Euclidean n-space R^n, f is a continuous mapping from cl (D) into R^n, and $q \notin f(\partial D)$. By ${}^*\mathbf{Z}$ we denote the ring of all equivalence classes $[s_n] = \{\{t_n\}: t_n = s_n$ for all n sufficiently large$\}$ of sequences of integers.

Definition 3. Let T: $(G) \to X$ be A-proper with respect to a given approximation scheme, where $G_n = G \cap X_n$ is bounded and open in X_n for all n sufficiently large and $T_n = P_n T|_{G_n}$ is continuous for all n sufficiently large. Let $y \in X \backslash T(\partial G)$. Then the degree of T at y relative to G is the element $D(T, G, y) = [s_n]$ of ${}^*\mathbf{Z}$ such that

$$s_n = \deg\,(T_n, G_n, P_n y)$$

for all n sufficiently large.

Note that $\deg(T_n, G_n, P_n y)$ is defined for all n sufficiently large since $P_n y \notin T_n(\partial G_n)$ for all n sufficiently large by [2, Lemma 1, p. 220]. The degree of Browder and Petryshyn [1], [2], Deg (T, G, y), is the set of limit points of $\{\deg\,(T_n, G_n, P_n y)\}$ including possibly $\pm\infty$.

In the following the Banach space l_2 of square summable real sequences with norm $\|(\alpha_i)\|^2 = \sum_{i=1}^{\infty} \alpha_i^2$ will have the oriented projectionally complete scheme $\Gamma(l_2)$ given by

$$X_n = \text{span}\,(e_1, e_2, \ldots, e_n) \quad \text{for } n = 1, 2, \ldots$$

(where e_i has coordinate 1 in the ith place and 0 elsewhere) and

$$P_n\left(\sum_{i=1}^{\infty} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i e_i \quad \text{for } n = 1, 2, \ldots.$$

The orientation of X_n is determined by the order $(e_1, e_2, \ldots, e_n)$ of the basis elements. Let H_m be the subset of l_2 given by

$$H_m = \{x \in l_2: \|me_m - x\| < \tfrac{1}{2}\}$$

and let G be given by

$$G = \bigcup_{m=1}^{\infty} H_m.$$

Then, as is shown in Appendix I, the H_m have disjoint closures, $\mathrm{cl}\,(H_m) \cap X_n$ is empty for all $m > n$, and $\mathrm{cl}\,(G) = \bigcup_{m=1}^{\infty} \mathrm{cl}\,(H_m)$.

THEOREM. *Given any element $[s_n]$ of $^*\mathbf{Z}$, there is a mapping T: $\mathrm{cl}\,(G) \to l_2$ which is A-proper with respect to $\Gamma(l_2)$ and such that $D(T, G, 0) = [s_n]$.*

PROOF. Let $t_0 = t_1 = 0$ and $t_n = s_n$ if $n \geqslant 2$. Then put

$$k_n = |t_n - t_{n-1}|, \quad \varepsilon_n = \mathrm{sign}(t_n - t_{n-1}),$$

and

$$a_n = 2n - 1, \quad b_n = 2n, \qquad n = 1, 2, \ldots.$$

Define the mapping T: $\mathrm{cl}\,(G) \to l_2$ as follows. For $x = \sum_{i=1}^{\infty} \alpha_i e_i \in \mathrm{cl}\,(G)$ there is a unique m such that $x \in \mathrm{cl}\,(H_m)$. Then put $T(x) = \sum_{i=1}^{\infty} \eta_i e_i$ where

$$\eta_{m-1} = \varepsilon_m \prod_{i=1}^{k_m} (\alpha_m - m - a_i \alpha_{m-1}),$$

$$\eta_m = \prod_{i=1}^{k_m} (\alpha_m - m - b_i \alpha_{m-1}),$$

$$\eta_i = \alpha_i \quad \text{for all } i \neq m-1, m,$$

for $m \geqslant 2$, and $\eta_i = \alpha_i$ for all i if $m = 1$.

First we show T is A-proper. Let $\{x_{n_j} \in G_{n_j}\}$ be a bounded sequence and $g \in l_2$ be such that $P_{n_j} T(x_{n_j}) \to g$. Let

$$x_{n_j} = \sum_{i=1}^{n_j} \alpha_{i,n_j} e_i, \quad P_{n_j} T(x_{n_j}) = \sum_{i=1}^{n_j} \beta_{i,n_j} e_i, \quad \text{and} \quad g = \sum_{i=1}^{\infty} \gamma_i e_i.$$

Since $\{x_{n_j}\}$ is bounded, there is a p such that $\{x_{n_j}\} \subseteq \bigcup_{m=1}^{p} \mathrm{cl}\,(H_m)$, for if $y \in \mathrm{cl}\,(H_m)$ then $\|y\| \geqslant \|me_m\| - \|y - me_m\| \geqslant m - \frac{1}{2}$. There is a subsequence of $\{x_{n_j}\}$ (call it $\{x_{n_j}\}$ again) and a $q \in \{1, \ldots, p\}$ such that $\{x_{n_j}\} \subseteq \mathrm{cl}\,(H_q)$.

Now $\{\alpha_{i,n_j}\}_{j=1}^{\infty}$ is bounded for each fixed i, since $|\alpha_{i,n_j}| \leqslant \frac{1}{2}$ for $i \neq q$ and $|\alpha_{q,n_j}| \leqslant q + \frac{1}{2}$. Hence there is a further subsequence $\{x_{n_{j(k)}}\}$ of $\{x_{n_j}\}$ and α_i ($i = 1, \ldots, q$) such that $\alpha_{i,n_{j(k)}} \to \alpha_i$ as $k \to \infty$, $i = 1, \ldots, q$. Put $\alpha_i = \gamma_i$ for $i > q$, and $x = \sum_{i=1}^{\infty} \alpha_i e_i$. Then

$$\begin{aligned}
\|x_{n_{j(k)}} - x\| &\leqslant \sum_{i=1}^{q} \|\alpha_{i,n_{j(k)}} e_i - \alpha_i e_i\| + \left\| \sum_{i=q+1}^{n_{j(k)}} \alpha_{i,n_{j(k)}} e_i - \sum_{i=q+1}^{\infty} \alpha_i e_i \right\| \\
&= \sum_{i=1}^{q} |\alpha_{i,n_{j(k)}} - \alpha_i| + \left\| \sum_{i=q+1}^{\infty} (\beta_{i,n_{j(k)}} - \gamma_i) e_i \right\| \\
&\leqslant \sum_{i=1}^{q} |\alpha_{i,n_{j(k)}} - \alpha_i| + \|P_{n_{j(k)}} T(x_{n_{j(k)}}) - g\|
\end{aligned}$$

which tends to zero as $k \to \infty$. Hence $x_{n_{j(k)}} \to x$ and $x \in l_2$. Indeed $x \in \mathrm{cl}\,(H_q)$ since $\{x_{n_{j(k)}}\} \subseteq \mathrm{cl}\,(H_q)$.

It remains to show that $T(x) = g$. Let $T(x) = \sum_{i=1}^{\infty} \eta_i e_i$. Assume $q > 1$.

Since $x \in \mathrm{cl}\,(H_q)$, $\eta_i = \alpha_i = \gamma_i$, $i \neq q-1, q$,

$$\eta_{q-1} = \varepsilon_q \prod_{i=1}^{k_q} (\alpha_q - q - a_i \alpha_{q-1}) = \lim_{k\to\infty} \varepsilon_q \prod_{i=1}^{k_q} (\alpha_{q,n_{j(k)}} - q - a_i \alpha_{q-1,n_{j(k)}})$$
$$= \lim_{k\to\infty} \beta_{q-1,n_{j(k)}} = \gamma_{q-1},$$

and

$$\eta_q = \prod_{i=1}^{k_q} (\alpha_q - q - b_i \alpha_{q-1}) = \lim_{k\to\infty} \prod_{i=1}^{k_q} (\alpha_{q,n_{j(k)}} - q - b_i \alpha_{q-1,n_{j(k)}})$$
$$= \lim_{k\to\infty} \beta_{q,n_{j(k)}} = \gamma_q .$$

For the case $q = 1$, since T restricted to $\mathrm{cl}\,(H_1)$ is the identity, $\eta_i = \alpha_i = \gamma_i$ for all $i = 1, 2, \ldots$. Hence in either case $T(x) = \sum_{i=1}^{\infty} \eta_i e_i = \sum_{i=1}^{\infty} \gamma_i e_i = g$, and T is A-proper with respect to $\Gamma(l_2)$.

We will now compute the degree of T at 0 relative to G. Since $H_m \cap X_n = \varnothing$ for $m > n$ (see Appendix I), it follows that

$$G_n = G \cap X_n = \bigcup_{m=1}^{n} H_{m,n} \quad \text{where } H_{m,n} = H_m \cap X_n .$$

Then

$$\deg\,(T_n, G_n, 0) = \sum_{m=1}^{n} \deg\,(T_n, H_{m,n}, 0)$$

by the sum formula for the Brouwer degree (see [3, Theorem 6.8, p. 32] or [8, Theorem 3.16.5, p. 72]) since the H_m, and thus the $H_{m,n}$, are disjoint for fixed n. Now, for $n \geqslant m \geqslant 2$, T_n is the identity on all components except the $(m-1)$st and mth. Thus, by the reduction formula for the Brouwer degree (see [3, Theorem 10.1, p. 51] or [8, Theorem 3.16.7, p. 72]),

$$\deg\,(T_n, H_{m,n}, 0) = \deg\,(T_{n,m}, H_{m,n} \cap E_m, 0),$$

where $E_m = \mathrm{span}\,(e_{m-1}, e_m)$ with orientation induced by the order of the basis elements, and $T_{n,m}$ is equal to T_n restricted to E_m. Let U_m be the translation on E_m given by

$$U_m(x) = x + me_m .$$

Then it follows (see Appendix II) that

$$\deg\,(T_{n,m}, H_{m,n} \cap E_m, 0) = \deg\,(T_{n,m} U_m, U_m^{-1}(H_{m,n} \cap E_m), 0).$$

Now

$$U_m^{-1}(H_{m,n} \cap E_m) = \{x \in E_m : \|x\| < \tfrac{1}{2}\}$$

and

$$T_{n,m} U_m(\alpha_{m-1} e_{m-1} + \alpha_m e_m) = \gamma_{m-1} e_{m-1} + \gamma_m e_m,$$

where

$$\gamma_{m-1} = \varepsilon_m \prod_{i=1}^{k_m} (\alpha_m - a_i \alpha_{m-1}),$$

and

$$\gamma_m = \prod_{i=1}^{k_m} (\alpha_m - b_i \alpha_{m-1}).$$

Hence, as shown by Cronin [3, pp. 38–40],

$$\deg (T_{n,m} U_m, U_m^{-1}(H_{m,n} \cap E_m), 0) = \varepsilon_m k_m.$$

Also T_n is the identity on $H_{1,n}$ and $0 \notin \mathrm{cl}\,(H_{1,n})$ so $\deg (T_n, H_{1,n}, 0) = 0$. Hence

$$\deg (T_n, G_n, 0) = \sum_{m=1}^{n} \deg (T_n, H_{m,n}, 0) = \sum_{m=1}^{n} \varepsilon_m k_m = t_n$$

since $t_1 = 0$. Thus

$$D(T, G, 0) = [t_n] = [s_n].$$

Appendix I.

1. We show that $\mathrm{cl}\,(H_m) \cap \mathrm{cl}\,(H_n) = \varnothing$ for $m \neq n$. If $x \in \mathrm{cl}\,(H_m)$ and $y \in \mathrm{cl}\,(H_n)$, then

$$\begin{aligned}\|x - y\| &\geqslant \|me_m - ne_n\| - \|me_m - x\| - \|ne_n - y\| \\ &\geqslant (m^2 + n^2)^{1/2} - \tfrac{1}{2} - \tfrac{1}{2} \geqslant 2^{1/2} - 1 > 0.\end{aligned}$$

Hence $x \neq y$ and $\mathrm{cl}\,(H_m) \cap \mathrm{cl}\,(H_n) = \varnothing$.

2. It also follows that $\mathrm{cl}\,(G) = \mathrm{cl}\,(\bigcup_{m=1}^{\infty} H_m) = \bigcup_{m=1}^{\infty} \mathrm{cl}\,(H_m)$. For if $\{g_n\} \subseteq G$ is such that $g_n \to x$ then there is an N such that $\|g_n - g_m\| \leqslant (2^{1/2} - 1)/2$ for all n, $m \geqslant N$. Hence there is a p such that $g_n \in H_p$ for all $n \geqslant N$. Hence $x \in \mathrm{cl}\,(H_p)$ and $\mathrm{cl}\,(G) \subseteq \bigcup_{m=1}^{\infty} \mathrm{cl}\,(H_m)$. Clearly $\bigcup_{m=1}^{\infty} \mathrm{cl}\,(H_m) \subseteq \mathrm{cl}\,(G)$, so $\mathrm{cl}\,(G) = \bigcup_{m=1}^{\infty} \mathrm{cl}\,(H_m)$.

3. We will now show that $\mathrm{cl}\,(H_m) \cap \mathrm{span}\,(e_1, \ldots, e_n) = \varnothing$ for all $m > n$. If $x \in \mathrm{cl}\,(H_m)$ and $y = \sum_{i=1}^{n} \alpha_i e_i \in \mathrm{span}\,(e_1, \ldots, e_n)$ and $m > n$ then

$$\|x - y\| = \left\| me_m - \sum_{i=1}^{n} \alpha_i e_i \right\| = \left(m^2 + \sum_{i=1}^{n} \alpha_i^2 \right)^{1/2} \geqslant m > 0.$$

Hence $x \neq y$ and $\mathrm{cl}\,(H_m) \cap \mathrm{span}\,(e_1, \ldots, e_n) = \varnothing$, for $m > n$.

Appendix II. Let D be an open bounded subset of R^n, f a continuous mapping from $\mathrm{cl}\,(D)$ to R^n and $q \notin f(\partial D)$. Let $x_0 \in X$ be fixed. Let U be the translation defined by $U(x) = x + x_0$. We will show that

$$\deg (fU, U^{-1}(D), q) = \deg (f, D, q).$$

By the product formula for the Brouwer degree [8, Theorem 3.20, p. 75]

$$\deg(fU, U^{-1}(D), q) = \sum_i \deg(f, B_i, q) \deg(U, U^{-1}(D), p_i),$$

where B_i are the bounded components of $R^n \backslash U(\partial U^{-1}(D)) = R^n \backslash \partial D$ and $p_i \in B_i$. But

$$\deg(U, U^{-1}(D), p_i) = \sum \operatorname{sign} |U'(x)|,$$

where $|U'(x)|$ is the determinant of the Jacobian matrix of U at x, and the sum is over all points x in $U^{-1}(p_i) \cap U^{-1}(D)$, i.e. $x = p_i - x_0$, $p_i \in B_i \cap D$. Now $U(x + h) - U(x) = h$ for all $x, h \in R^n$, so $U'(x)$ is the identity matrix and $|U'(x)| = 1$. Hence,

$$\deg(fU, U^{-1}(D), q) = \sum_i \deg(f, B_i, q),$$

where the sum is over the bounded components of D. Thus by the sum formula for the Brouwer degree (see [3, Theorem 6.8, p. 32] or [8, Theorem 3.16.5, p. 72]), $\deg(fU, U^{-1}(D), q) = \deg(f, D, q)$.

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