PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

# A GENERALIZATION OF THE HAHN-MAZURKIEWICZ THEOREM

L. E. WARD, JR.

ABSTRACT. It is proved that if a Hausdorff continuum $X$ can be approximated by finite trees (see the text for definition) then there exists a (generalized) arc $L$ and a continuous surjection $\varphi: L \to X$.

1. **Introduction**. The celebrated Hahn-Mazurkiewicz theorem, first proved about 1914 [4], [8], asserts that a Peano continuum is the image of [0, 1] under some continuous mapping. Subsequent attempts to generalize the theorem to the nonmetric setting proved unavailing, and in 1960 Mardešić [6] described a locally connected Hausdorff continuum which is not arcwise connected (in the generalized sense) and hence is not the continuous image of any arc. Later Cornette and Lehman [3] exhibited a simpler example with the same properties. The possibility remained that an arcwise connected, locally connected continuum is the continuous image of some arc, but in [7] Mardešić and Papić showed that any product of continua which is the continuous image of an arc is necessarily metrizable. Consequently, even such a nice continuum as $L \times [0, 1]$, where $L$ is the "long arc", is not the continuous image of an arc. Later results of Treybig [12], [13], A. J. Ward [15] and Young [19] elaborated on this theme.

Quite recently some affirmative results have appeared. Cornette [2] proved that a tree is the continuous image of some arc, and the author [17] has extended this to rim-finite continua. Different proofs of these results have been found independently by Pearson [10], [11].

In this paper we prove a generalization of the Hahn-Mazurkiewicz theorem which includes all of the aforementioned affirmative results.

We recall some terminology. A *continuum* is a compact, connected Hausdorff space. An *arc* is a continuum with exactly two noncutpoints. A *tree* is a continuum in which each pair of distinct points can be separated by some point. A *finite tree* is a tree with only finitely many endpoints.

A continuum $X$ can be *approximated by finite trees* if there exists a family $\mathfrak{T}$ of finite trees such that

(1) $\mathfrak{T}$ *is directed by inclusion*,

(2) $\bigcup \mathfrak{T}$ *is dense in* $X$,

(3) *if* $\mathfrak{U}$ *is an open cover of* $X$ *then there exists* $T(\mathfrak{U}) \in \mathfrak{T}$ *such that if*

Received by the editors June 6, 1975.

*AMS (MOS) subject classifications* (1970). Primary 54C05, 54F25; Secondary 54B25, 54D05.

*Key words and phrases*. Hahn-Mazurkiewicz theorem, continuum, arc, continuous image of an arc, approximation by finite trees, approximation by a sequence of finite dendrites, inverse limit.

*$T(\mathfrak{U}) \subset T \in \mathfrak{T}$, and if $C$ is a component of $T - T(\mathfrak{U})$, then there exists $U \in \mathfrak{U}$ such that $C \subset U$.*

Our principal result is the following.

THEOREM 1. *If $X$ is a continuum which can be approximated by finite trees then there exists an arc $L$ and a continuous surjection $\varphi: L \to X$.*

## 2. **Proof of Theorem 1.**

LEMMA 1. *If $\{T_\alpha, r_{\beta\alpha}\}$ is an inverse system of trees and if the bonding mappings $r_{\beta\alpha}$ are monotone, then $T_\infty = \text{inv lim}\{T_\alpha, r_{\beta\alpha}\}$ is a tree.*

PROOF. Nadler [9, Theorem 3] has shown that $T_\infty$ is hereditarily unicoherent, and Capel [1] proved that $T_\infty$ is locally connected. Hence [16, Theorem 9], $T_\infty$ is a tree.

LEMMA 2. *If $T_1$ and $T_2$ are trees with $T_1 \subset T_2$, then there exists a retraction $r: T_2 \to T_1$ which is monotone. Moreover, if $C$ is a component of $T_2 - T_1$ then $C$ has one-point boundary $x(C)$ and $r(C) = x(C)$.*

PROOF. If $C$ is a component of $T_2 - T_1$ then, by the hereditary unicoherence of trees, $\overline{C} \cap T_1$ is connected. Suppose $\overline{C} \cap T_1$ contains distinct elements $x$ and $y$; then there are connected neighborhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $\overline{U}_x$ and $\overline{U}_y$ are disjoint. Since $C$ is an open set, we can invoke a standard chaining argument to show the existence of a continuum $K$ which is contained in $C$ and which meets both $U_x$ and $U_y$. If we define $P = \overline{U}_x \cup K \cup \overline{U}_y$ and $Q = \overline{C} \cap T_1$, then $P$ and $Q$ are subcontinua of $T_2$, $P \cap Q \subset (\overline{U}_x \cup \overline{U}_y)$, and $P \cap Q$ meets both $\overline{U}_x$ and $\overline{U}_y$. This contradicts the hereditary unicoherence of the tree $T_2$, and hence $\overline{C} \cap T_1 = \overline{C} - C$ consists of a single point, $x(C)$. Define $r: T_2 \to T_1$ by $r|T_1 = 1$ and $r(C) = x(C)$ for each component $C$ of $T_2 - T_1$. It is straightforward to verify that $r$ is continuous. Finally, $r$ is monotone because, for each $x \in T_2$,

$$r^{-1}(x) = \{x\} \cup \bigcup \left\{C\colon C \textit{ is a component of } T_2 - T_1 \textit{ and } \overline{C} \cap T_1 = \{x\}\right\},$$

which is a connected set.

For the remainder of this section let $X$ be a continuum which is approximated by the family $\mathfrak{T}$ of finite trees. Then the system $\mathfrak{T} = \{T_\alpha, r_{\beta\alpha}\}$ is an inverse system with monotone bonding maps, and hence $T_\infty = \text{inv lim } \mathfrak{T}$ is a tree.

LEMMA 3. *If $(x_\alpha) \in T_\infty$ then $(x_\alpha)$ is a convergent net in $X$.*

PROOF. Let $p$ be a cluster point of the net $(x_\alpha)$ and suppose $V$ is an open set containing $p$. There exists a finite open cover $\beta$ of $X$ such that if $p \in U \in \beta$ then $\text{Star}(U, \beta) \subset V$. By hypothesis there exists $T_\beta \in \mathfrak{T}$ such that if $T_\beta \subset T_\gamma \in \mathfrak{T}$ and if $C$ is a component of $T_\gamma - T_\beta$, then $C$ lies in some member of $\beta$; moreover, we may assume $x_\beta \in U$. If $x_\beta \neq x_\gamma$ then, since $r_{\gamma\beta}(x_\gamma) = x_\beta$, it follows that the component $C$ of $T_\gamma - T_\beta$ which contains $x_\gamma$ has $\{x_\beta\}$ for boundary and hence $C \subset \text{Star}(U, \beta) \subset V$. Therefore the net $(x_\alpha)$ converges to $p$.

LEMMA 4. *The function* $g\colon T_\infty \to X$ *defined by* $g((x_\alpha)) = \lim(x_\alpha)$ *is a continuous surjection.*

PROOF. Let $p = \lim(x_\alpha)$ and suppose $V$ is an open set containing $p$. Choose a finite open cover $\beta$ of $X$ and $T_\beta \in \mathfrak{T}$ as in Lemma 3. If $p \in U \in \beta$, let $W = \pi_\beta^{-1}(U \cap T_\beta) \cap T_\infty$, a neighborhood of $(x_\alpha)$ in $T_\infty$ ($\pi_\beta$ denotes the projection function). If $(y_\alpha) \in W$ then $y_\beta \in U$ and hence, if $T_\beta \subset T_\gamma \in \mathfrak{T}$, it follows that $y_\gamma \in \operatorname{Star}(U, \beta) \subset V$. Therefore $g((y_\alpha)) \in \overline{V}$ and so $g$ is continuous.

To see that $g$ is surjective let $(x_\alpha) \in T_\infty$ with $(x_\alpha)$ eventually constant. That is, there exists $T_\beta \in \mathfrak{T}$ such that $x_\gamma = x_\beta$ for all $T_\gamma \in \mathfrak{T}$ with $T_\beta \subset T_\gamma$. Then $g((x_\alpha)) = x_\beta$ and hence $g(T_\infty) \supset \bigcup \mathfrak{T}$. Since $g$ is continuous and $\bigcup \mathfrak{T}$ is dense in $X$ it follows that $g(T_\infty) = X$.

PROOF OF THEOREM 1. By [2] and Lemma 1 there is an arc $L$ and a continuous surjection $f\colon L \to T_\infty$. By Lemma 4 the function $\varphi = gf\colon L \to X$ is the desired mapping.

Recently E. D. Tymchatyn [14] has applied Theorem 1 to prove that each finitely Suslinian Hausdorff continuum is the continuous image of an arc. This generalizes the result of Cornette, Pearson and the author [2], [10], [11], [17] for trees and rim-finite continua.

It is irresistible to inquire whether the condition of being approximated by finite trees is necessary as well as sufficient for a continuum to be the continuous image of an arc. I conjecture that the answer is affirmative.

3. **The classical Hahn-Mazurkiewicz theorem**. Recall that a *dendrite* is a metrizable tree. In attempting to deduce the classical theorem from Theorem 1, we consider a metric continuum $M$. We wish to show that if $M$ can be approximated by a sequence of finite dendrites then $M$ is the continuous image of $[0, 1]$. It follows from Theorem 1 that $M$ is the image of some arc, but we have no assurance that the arc is separable. The proof that $M$ is the continuous image of $[0, 1]$ is facilitated by the following two lemmas.

LEMMA 5. *If $D$ is a finite dendrite then there exists a continuous surjection* $f\colon [0, 1] \to D$.

PROOF. Since $D$ has only a finite set $\{e_1, \dots, e_n\}$ of endpoints, $n \geqslant 2$, we may write $D = A_2 \cup \cdots \cup A_n$ where $A_2 = [e_1, e_2]$ is an arc and $A_k = [d_k, e_k]$ is an arc irreducible between $(A_1 \cup \cdots \cup A_{k-1})$ and $e_k$ where $2 < k \leqslant n$. There is a homeomorphism $f_2\colon [0, 1] \to A_2$; suppose $f_{k-1}\colon [0, 1] \to (A_1 \cup \cdots \cup A_{k-1})$ is a continuous surjection with $f_{k-1}(t) = d_k$. Without loss of generality we may assume $0 < t < 1$. Define

$$h_1\colon [0, t] \to [0, \tfrac{1}{4}] \quad \text{by } h_1(x) = x/4t,$$

$$h_2\colon [t, 1] \to [\tfrac{3}{4}, 1] \quad \text{by } h_2(x) = (x + 3 - 4t)/4(1 - t).$$

Let

$$g_1\colon [\tfrac{1}{4}, \tfrac{1}{2}] \to [d_k, e_k] \quad \text{and} \quad g_2\colon [\tfrac{1}{2}, \tfrac{3}{4}] \to [e_k, d_k]$$

be homeomorphisms which preserve the indicated endpoints. If we define

$$f_k = \begin{cases} f_{k-1}h_1^{-1} & \textit{on } \left[0, \frac{1}{4}\right], \\ g_1 & \textit{on } \left[\frac{1}{4}, \frac{1}{2}\right], \\ g_2 & \textit{on } \left[\frac{1}{2}, \frac{3}{4}\right], \\ f_{k-1}h_2^{-1} & \textit{on } \left[\frac{3}{4}, 1\right], \end{cases}$$

then $f_k$: $[0, 1] \to (A_1 \cup \cdots \cup A_k)$ is a continuous surjection, and the lemma follows by induction.

LEMMA 6. *If $D$ and $D'$ are finite dendrites with $D \subset D'$, $r$: $D' \to D$ is the natural monotone retraction and $f$: $[0, 1] \to D$ is a continuous surjection, then there exists a monotone mapping $s$: $[0, 1] \to [0, 1]$ and a continuous surjection $f'$: $[0, 1] \to D'$ such that $fs = rf'$.*

PROOF. There are only finitely many elements $x_1, \ldots, x_n$ of $D$ which are the boundaries of components of $D' - D$. For each $i = 1, \ldots, n$ let

$$K_i = \{x_i\} \cup \bigcup \left\{C\text{: } C \textit{ is a component of } D' - D \textit{ and } x_i \in \bar{C}\right\},$$

and choose $t_i \in f^{-1}(x_i)$. Without loss of generality we assume $0 < t_1 < t_2 < \cdots < t_n < 1$. Define linear homeomorphisms $h_0, \ldots, h_n$ as follows:

$$h_0\colon [0, t_1] \to [0, 1/(2n + 1)] \quad \textit{by } h_0(x) = x/(2n + 1)t_1,$$

$$h_k\colon [t_k, t_{k+1}] \to [2k/(2n + 1), (2k + 1)/(2n + 1)]$$
$$\textit{by } h_k(x) = (x + 2kt_{k+1} - (2k + 1)t_k)/(2n + 1)(t_{k+1} - t_k),$$
$$k = 1, \ldots, n - 1,$$

$$h_n\colon [t_n, 1] \to \left[\frac{2n}{2n + 1}, 1\right] \quad \textit{by } h_n(x) = \frac{x + 2n - (2n + 1)t_n}{(2n + 1)(1 - t_n)}.$$

Each of the sets $K_i$ is a finite dendrite, so by Lemma 5 there is a continuous surjection

$$g_i\colon [(2i - 1)/(2n + 1), 2i/(2n + 1)] \to K_i, \qquad i = 1, \ldots, n.$$

Define $s$: $[0, 1] \to [0, 1]$ by

$$s = h_{i-1}^{-1} \quad \textit{on } [(2i - 2)/(2n + 1), (2i - 1)/(2n + 1)], \qquad 1 \leqslant i \leqslant n + 1,$$
$$s(t) = t_i \quad \textit{if } t \in [(2i - 1)/(2n + 1), 2i/(2n + 1)], \qquad 1 \leqslant i \leqslant n,$$

and define $f'$: $[0, 1] \to D'$ by

$$f' = \begin{cases} fh_{i-1}^{-1} \quad \textit{on } [(2i - 2)/(2n + 1), (2i - 1)/(2n + 1)], \\ \qquad\qquad\qquad\qquad 1 \leqslant i \leqslant n + 1, \\ g_i \quad \textit{on } [(2i - 1)/(2n + 1), 2i(2n + 1)], \qquad 1 \leqslant i \leqslant n. \end{cases}$$

Then it is obvious that $s$ is continuous and monotone, that $f'$ is a continuous surjection and that $fs = rf'$.

We say that a metric continuum $M$ can be *approximated by a sequence of finite dendrites* if there exists a sequence $D_1, D_2, \ldots, D_n, \ldots$ of finite dendrites such that

(1) $D_1 \subset D_2 \subset \cdots \subset D_n \subset \ldots,$
(2) $\bigcup\{D_n\colon n = 1, 2, \ldots\}$ *is dense in* $M$,
(3) *if* $C$ *is a component of* $D_{n+1} - D_n$ *then* $\operatorname{diam}(C) < 2^{-n}$.

THEOREM 2. *If $M$ is a metric continuum then the following statements are equivalent*:
(i) *there exists a continuous surjection* $\psi\colon [0, 1] \to M$,
(ii) *$M$ is a Peano continuum*,
(iii) *$M$ can be approximated by a sequence of finite dendrites*.

PROOF. It is well known that (i) $\Rightarrow$ (ii). (For example, consult [5].)

To see that (ii) $\Rightarrow$ (iii), it is a consequence of the fact that $M$ is compact and locally connected that $M$ admits a sequence $\mathcal{U}_n$ of finite connected open covers such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ and $\operatorname{diam}(U) < 2^{-n}$ for each $U \in \mathcal{U}_n$. Independent of the Hahn-Mazurkiewicz theorem it can be shown that each member of $\mathcal{U}_n$ is arcwise connected. (See [18, Chapter II, §5, under the second remark on p. 39, together with 5.3].) Therefore it is possible to construct a sequence of finite dendrites $D_1, D_2, \ldots$ such that $D_n$ meets each member of $\mathcal{U}_n$, $D_n \subset D_{n+1}$, and each component of $D_{n+1} - D_n$ lies in some member of $\mathcal{U}_n$.

To prove (iii) $\Rightarrow$ (i), let $M$ be approximated by the sequence $D_1 \subset D_2 \subset \ldots$ of finite dendrites. By Lemmas 5 and 6 there are continuous surjections $f_n$ and continuous monotone surjections $r_n$ and $s_n$ so that the ladder

$$\begin{tikzcd}
D_1 & D_2 \arrow[l, "r_1"'] & \cdots \arrow[l, "r_2"'] & D_n \arrow[l, "r_{n-1}"'] & \cdots \arrow[l, "r_n"'] \\
{[0,1]} \arrow[u, "f_1"] & {[0,1]} \arrow[l, "s_1"'] \arrow[u, "f_2"] & \cdots \arrow[l, "s_2"'] & {[0,1]} \arrow[l, "s_{n-1}"'] \arrow[u, "f_n"] & \cdots \arrow[l, "s_n"']
\end{tikzcd}$$

is commutative. It follows that $D_\infty = \operatorname{inv}\lim\{D_n, r_n\}$ is a dendrite, the limit of the inverse sequence $\{[0, 1], s_n\}$ is $[0, 1]$, and there is induced a continuous surjection $f\colon [0, 1] \to D_\infty$. Lemmas 3 and 4 now apply and hence there is a continuous surjection $g\colon D_\infty \to M$. Let $\psi = gf\colon [0, 1] \to M$.

## REFERENCES


1. C. E. Capel, *Inverse limit spaces*, Duke Math. J. **21** (1954), 233–245. MR **15**, 976.

2. J. L. Cornette, *Image of a Hausdorff arc is cyclically extensible and reducible*, Trans. Amer. Math. Soc. **199** (1974), 253–267.

3. J. L. Cornette and B. Lehman, *Another locally connected Hausdorff continuum not connected by ordered continua*, Proc. Amer. Math. Soc. **35** (1972), 281–284. MR **46** #6320.

4. H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurven*, Sitzungsber. Akad. Wiss. Wien **123** (1914), 2433–2489.

5. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961. MR **23** #A2857.

6. S. Mardešić, *On the Hahn-Mazurkiewicz theorem in nonmetric spaces*, Proc. Amer. Math. Soc. **11** (1960), 929–937. MR **22** #8464.

7. S. Mardešić and P. Papić, *Continuous images of ordered continua*, Glasnik Mat.-Fiz. Astronom. Društro Mat. Fiz. Hrvatske Ser. II **15** (1960), 171–178. MR **24** #A536.

8. S. Mazurkiewicz, *Sur les lignes de Jordan*, Fund. Math. **1** (1920), 166–209.

9. S. B. Nadler, Jr., *Multicoherence techniques applied to inverse limits*, Trans. Amer. Math. Soc. **157** (1971), 227–234. MR **43** #5482.

10. B. J. Pearson, *Mapping an arc onto a dendritic continuum*, Colloq. Math. **30** (1974), 237–243.

11. ———, *Mapping arcs and dendritic spaces onto netlike continua*, Colloq. Math. **34** (1975), 39–48.

12. L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. **15** (1964), 866–871. MR **29** #5218.

13. ———, *Concerning continua which are continuous images of compact ordered spaces*, Duke Math. J. **32** (1965), 417–422. MR **32** #4673.

14. E. D. Tymchatyn, *The Hahn-Mazurkiewicz theorem for finitely Suslinian continua* (to appear).

15. A. J. Ward, *Notes on general topology*. III, *A non-metric image of an ordered compactum*, Proc. Cambridge Philos. Soc. **61** (1965), 881–882. MR **32** #1685.

16. L. E. Ward, Jr., *Mobs, trees and fixed points*, Proc. Amer. Math. Soc. **8** (1957), 798–804. MR **20** #3516.

17. ———, *The Hahn-Mazurkiewicz theorem for rim-finite continua*, General Topology and Appl. (to appear).

18. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R. I., 1942. MR **4**, 86.

19. G. S. Young, Jr., *Representations of Banach spaces*, Proc. Amer. Math. Soc. **13** (1962), 667–668. MR **26** #574.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403