PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

3-MANIFOLDS FIBERING OVER S^1

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ABSTRACT. Let M be a closed 3-manifold that is the total space of a fiber bundle with base S^1 and fiber the closed 2-manifold F. Assume that genus $(F) \geqslant 2$ if F is orientable, and that genus $(F) \geqslant 3$ if F is nonorientable. We say that M has *unique fiber over* S^1 if, for any fibering of M over S^1 with fiber F', we have $F' \cong F$. We prove that M has unique fiber over S^1 if and only if rank $(H_1(M; \mathbf{Z})) = 1$. In the case that rank $(H_1(M; \mathbf{Z})) \neq 1$, M fibers over S^1 with fiber any of infinitely many distinct closed surfaces.

In [5], Tollefson proved that if M is a 3-manifold of the form $F \times S^1$, where F is a closed oriented surface of genus $g \geqslant 2$, then M fibers over S^1 with fiber any of infinitely many distinct surfaces. We extend this result to a characterization of uniqueness of the fiber in 3-manifolds fibering over S^1.

All manifolds and maps considered will be differentiable (say C^1). We say that the closed 3-manifold M *fibers over* S^1 *with fiber* F if M is diffeomorphic to the quotient manifold $(F \times I)/d$ obtained by identifying $F \times \{0\} \subseteq F \times I$ with $F \times \{1\}$ under the diffeomorphism $d: F \to F$. We say that the fiber is *unique* if in any other fibering $M \cong (F' \times I)/d'$ we have $F' \cong F$. Our result is then:

THEOREM. *Suppose M is a closed 3-manifold that fibers over S^1 with fiber a surface F of genus $g \geqslant 2$ ($g \geqslant 3$ if F is nonorientable); then the fiber is unique if and only if* rank $(H_1(M; \mathbf{Z})) = 1$.

PROOF. To prove the sufficiency of the rank condition suppose that M fibers as $(F \times I)/d$ and as $(F' \times I)/d'$, with $F' \not\cong F$. Note that $\pi_1(M)$ decomposes as a semidirect product $\pi_1(F) \rtimes_{d_*} \mathbf{Z}$ (the action of a generator z of the infinite cyclic factor on the normal subgroup $\pi_1(F)$ is given by $zfz^{-1} = d_*(f)$, for all $f \in \pi_1(F)$) and also as $\pi_1(F') \rtimes_{d'_*} \mathbf{Z}$. Now assume that rank $(H_1(M; \mathbf{Z})) = 1$. Then, since z maps to the generator of an infinite cyclic direct summand of $H_1(M; \mathbf{Z})$ under the abelianizing (Hurewicz) homomorphism, one can easily prove that $\pi_1(F) \subseteq \pi_1(M)$ consists of just those elements that are torsion modulo the commutator subgroup of $\pi_1(M)$. The same holds for $\pi_1(F') \subseteq \pi_1(M)$, and hence we see that $\pi_1(F)$ is isomorphic to $\pi_1(F')$. This contradiction shows that we must have rank $(H_1(M; \mathbf{Z})) \geqslant 2$.

To prove the converse we assume that $M = (F \times I)/d$ and that rank $(H_1(M; \mathbf{Z})) \geqslant 2$. To find a fiber $F_n \subseteq M$ distinct from F we first construct a map P: $M \to S^1 \times S^1$; F_n will then be realized as the inverse

Received by the editors May 19, 1975.
AMS (MOS) subject classifications (1970). Primary 57A10; Secondary 55A99.

image of a 1-submanifold $W_n \subseteq S^1 \times S^1$ on which P is transverse.

We first show that there is an epimorphism $\pi : \pi_1(F) \to \mathbf{Z}$ satisfying $\pi d_*^{-1} = \pi$. In the commutative diagram, β is the projection with kernel $\pi_1(F)$, α is the Hurewicz homomorphism, and $\bar{\beta}$ is induced by β.

$$\begin{tikzcd}
\pi_1(F) \arrow[r, "\gamma = \alpha|"] \arrow[d] & \ker \bar{\beta} \arrow[r] \arrow[d] & \mathbf{Z} \\
\pi_1(F) \rtimes_{d_*} \mathbf{Z} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & H_1(M; \mathbf{Z}) \arrow[d, "\bar{\beta}"] & \\
\mathbf{Z} \arrow[r, "1"] & \mathbf{Z} &
\end{tikzcd}$$

Note that $\gamma = \alpha|_{\pi_1(F)}$ is onto. Also, we may choose a generator $z \in \pi_1(M)$ of the infinite cyclic factor so that $d_*^{-1}(x) = z^{-1}xz$, for all $x \in \pi_1(F)$; hence

$$\gamma d_*^{-1}(x) = \gamma(z^{-1}xz) = \alpha(z^{-1}xz) = \alpha(x) = \gamma(x).$$

Since $\ker \bar{\beta}$ has at least one infinite cyclic summand by our assumption on $H_1(M; \mathbf{Z})$, we may define π to be the composition of γ with the projection onto such a summand.

Next, there is a differentiable map $p : F \to S^1$ realizing π; i.e., with $p_* = \pi$. By Lemma 3.2 of Jaco [2] we may assume that p is transversal with respect to a point $s_0 \in S^1$, and that, for each s in some open interval about s_0, $p^{-1}(s)$ is a simple closed curve in F. Since $p_* d_*^{-1} = p_*$, there is a homotopy p_t ($t \in [0, 1]$) of p to pd^{-1}; we may assume that this homotopy defines a differentiable map $F \times I \to S^1$ (cf. Hu [1, Lemma 2]). We will also assume that $p_t = p$, for $t \in [0, 1/3]$ say, and that $p_t = pd^{-1}$, for $t \in [2/3, 1]$. It follows that we may define a differentiable map

$$P : (F \times I)/d \to S^1 \times S^1 \quad (= (S^1 \times I)/1)$$

by $P(x, t) = (p_t(x), t)$.

We now want to find a 1-submanifold $W_n \subseteq S^1 \times S^1$ so that P is transverse on W_n. We consider local coordinates of the form

$$(x_1, x_2, t) \in (F \times I)/d$$

(where $x = (x_1, x_2)$ are local coordinates on F and $t \in J(\text{open}) \subseteq I$) and $(s, t) \in S^1 \times S^1$. Then the derivative DP of P is represented by the matrix

$$\begin{bmatrix} \dfrac{\partial p_t}{\partial x_1} & \dfrac{\partial p_t}{\partial x_2} & \dfrac{\partial p_t}{\partial t} \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the bases $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial t\} \subseteq T(M)_{(x,t)}$ and $\{\partial/\partial s, \partial/\partial t\} \subseteq T(S^1 \times S^1)_{P(x,t)}$. Thus the image of the tangent vector $(0, 0, 1)$ (i.e., $\partial/\partial t$) in $T(M)_{(x,t)}$ is $(\partial p_t/\partial t, 1) \in T(S^1 \times S^1)_{P(x,t)}$. We want to choose $W_n \subseteq S^1 \times S^1$ so that, at $w \in W_n$, $T(W_n)_w$ does not contain any vector of the form $DP_{(x,t)}(\partial/\partial t)$ ($(x, t) \in P^{-1}(w)$). If we fix finite covers of M and

$S^1 \times S^1$ consisting of charts of the above form, then

$$m = \sup_M |\partial p_t / \partial t|$$

(supremum over all local representations of p_t in terms of these fixed charts) will be finite. A sufficient condition that P be transverse on W_n is that if $a\partial/\partial s + b\partial/\partial t \in T(W_n)$, then $0 < b/a < 1/m$. It is clear that, for all sufficiently large n, there is such a simple closed curve W_n that winds n times around the $S^1 \times \{0\}$ factor of $S^1 \times S^1$ and once around the $\{s_0\} \times S^1$ factor.

Fix such a W_n and define $F_n \subseteq M$ to be the component of $P^{-1}(W_n)$ that contains $p^{-1}(s_0)$. Then F_n is a codimension one (differentiable) submanifold of M (cf. [3, pp. 21–25]).

To see that M fibers as $(F_n \times I)/d_n$, we use the fact that F_n is a *section* in the sense of Smale ([2, p. 99]) for the ' natural' flow ϕ on M determined by the diffeomorphism d. The flow $\phi : M \times \mathbf{R}^1 \to M$ is defined as follows: M may be realized as the quotient space of $F \times \mathbf{R}^1$ under the equivalence relation $(x, t) \sim (dx, t + 1)$; ϕ is then the flow induced by the constant vector field $(0, 0, 1)$ on $F \times \mathbf{R}^1$. The fact that $DP(0, 0, 1)$ is not in $T(W_n)$ implies that F_n is transverse to ϕ. To show that F_n is a (complete) section for ϕ it remains to show that, for any point $(x, t) \in F_n$, there is a time $\tau > 0$ so that $\phi(x, t, \tau) \in F_n$ (cf. remark following the definition of *section*, p. 99 of [4]). To see this, let $s_{0,0} = (s_0, 0)$, $s_{0,1} = (s_0, t_1), \ldots, s_{0,n-1} = (s_0, t_{n-1})$ denote the successive intersections of W_n with $\{s_0\} \times S^1$, let $(s_{0,0}, s_{0,1}]$ denote the half-open segment of W_n running from $s_{0,0}$ to $s_{0,1}$, and let $F_{n,1}$ denote $P^{-1}((s_{0,0}, s_{0,1}])$. We may assume that $(s_{0,0}, s_{0,1}] \subseteq S^1 \times [0, 1/3]$; in this case Int $(F_{n,1})$ is mapped homeomorphically onto $F - (p^{-1}(s_0))$ under the projection of $F \times I$ onto the first factor. Thus we may define a (height) function $h : F \to I$ by $h(x) = t$ if $(x, t) \in F_{n,1}$ (h is continuous except at $p^{-1}(s_0)$). Then if $(x, t) \in F_n$ we have $1 - t + h(d^{-1}x) > 0$ and $\phi(x, t, 1 - t + h(d^{-1}x)) \in F_n$ as required. It now follows by Theorem 2.2 of [4] that $M \cong (F_n \times I)/d_n$, where d_n is the diffeomorphism of F_n induced by ϕ.

To complete the proof we must show that $F_n \ncong F$. We show, in fact, that we can obtain surfaces F_n of arbitrarily high genus by varying n. Suppose that $[s_{0,0}, s_{0,1}], \ldots, [s_{0,k-1}, s_{0,k}]$ (notation as above) are all contained in $S^1 \times [0, 1/3]$, and let

$$F_{n,i} = P^{-1}([s_{0,i-1}, s_{0,i}]), \qquad i = 1, 2, \ldots, k,$$

$$F_{n,0} = P^{-1}(\mathrm{cl}\,(W_n - [s_{0,0}, s_{0,k}])).$$

Then each $F_{n,i}$ is a compact surface with two boundary components and, for $i = 1, 2, \ldots, k$, we have $F_{n,i} \cong F - N(p^{-1}(s_0))$ (here $N(p^{-1}(s_0))$ denotes a regular neighborhood of $p^{-1}(s_0)$). Thus

$$\chi(F_{n,0}) \leqslant 0,$$

$$\chi(F_{n,i}) = 2 - 2g \qquad (i = 1, 2, \ldots, k, F \text{ orientable}) \text{ or}$$

$$\chi(F_{n,i}) \leqslant 2 - g \qquad (i = 1, 2, \ldots, k, F \text{ nonorientable}).$$

It follows that $\chi(F_n) \leqslant k(2 - 2g)$ ($\chi(F_n) \leqslant k(2 - g)$ in case F is nonorientable) is arbitrarily large negative, as asserted.

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