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NEWMAN'S THEOREM FOR COMPACT RIEMANNIAN MANIFOLDS

MEI-CHIN KU

ABSTRACT. M. H. A. Newman has shown the following theorem: *Let M be a connected topological manifold with a given metric. Then there is an $\varepsilon > 0$ such that for every nontrivial action of a compact Lie group G on M, there exists an orbit of diameter at least ε.* We obtain the best possible estimate of ε for the isometric actions of G on compact connected Riemannian manifolds.

1. **Introduction.** Let M be a connected Riemannian manifold with Riemannian metric g which induces a metric d on M. Assume the group of isometries $I(M)$ of M is nontrivial. Define the *equivariant diameter* $D(M)$ of M by

$$D(M) = \sup\left\{\varepsilon \,\middle|\, \begin{array}{l}\text{every isometric action of a compact Lie group } G \text{ on} \\ M \text{ with } d(x,fx) < \varepsilon \text{ for all } x \in M, f \in G \text{ is trivial}\end{array}\right\}.$$

In the category of connected Riemannian manifolds, Newman's Theorem states that $D(M) \geqslant \varepsilon$, for some $\varepsilon > 0$ [2], [3]. That is, for every nontrivial isometric action of a compact Lie group G on M, there exists an orbit of diameter at least ε. Let $C(X)$ denote the cut locus of x [6], and define

$$c(M) = \inf\{d(x, C(x))|x \in M\}.$$

It is well known that $c(M) > 0$ if M is compact [8]. The purpose of this paper is to estimate $D(M)$ of a compact connected Riemannian manifold M in terms of the invariant $c(M)$. Moreover, if the compact manifold M has everywhere positive sectional curvature K, we also estimate the equivariant diameter $D(M)$ in terms of the upper bound of sectional curvature K of M. The main results are as follows:

THEOREM 1. *Suppose M is a compact connected Riemannian manifold. Then we have the inequality $D(M) \geqslant 2c(M)/3$.*

THEOREM 2. *Let M be a compact connected Riemannian manifold with sectional curvature K satisfying the inequality*

$$(*) \qquad 0 < h \leqslant K \leqslant H \quad \textit{for some constants } h \textit{ and } H.$$

Then

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$$D(M) \geqslant 2\pi/3|\pi_1(M)|\sqrt{H},$$

where $|\pi_1(M)|$ *denotes the order of the fundamental group* $\pi_1(M)$. (*Here the group* $\pi_1(M)$ *is finite* [6, p. 88].)

THEOREM 3. *Suppose* M *is a compact connected Riemannian manifold with negative Ricci tensor and with sectional curvature* $K \leqslant 0$. *Then we have* $D(M) \geqslant c(M)$.

REMARK. If M is a compact connected Riemannian manifold with sectional curvature $K < 0$, then the hypothesis of Theorem 3 is satisfied.

The estimates in both Theorems 1 and 2 are best possible in the sense that $D(S^n) = 2c(S^n)/3 = 2\pi/3$ and $D(P_n(R)) = \pi/3$, where S^n is the unit sphere in R^{n+1} with the usual geodesic metric, and $P_n(R)$ the real projective n-space covered by S^n because both S^n and $P_n(R)$ have constant sectional curvature 1 [5, p. 294]. In [9], Mann and Sicks also estimate $D(M)$ in terms of the radius of convexity $r(M)$ (for definition see [9]) of the compact connected Riemannian manifold M. They prove that $D(M) \geqslant r(M)$ which is much weaker than our results as we can see from the example $D(S^n) \geqslant r(S^n) = \pi/4 \quad (n \geqslant 2)$.

We shall use the notation and terminology of [5], [6] and [7]. The author wishes to thank Professors J. A. Wolf and V. Ozols for helpful comments in revising the original manuscript of this paper.

2. **Proofs of the theorems.** The following result is essentially due to Frankel (cf. [7, p. 57]).

LEMMA 1. *Let* M *be a compact Riemannian manifold with negative definite Ricci tensor and with the sectional curvature* $K \leqslant 0$, *and let* $\tilde{M}$ *be the universal covering space of* M *with covering projection* π. *If an isometry* f *of* M *has a unique lifting* $\tilde{f}$ *which is an isometry of* $\tilde{M}$ *so that for every* $\tilde{x} \in \tilde{M}$, *the distance* $d(\tilde{x},\tilde{f}(\tilde{x}))$ *depends only on* $\pi(\tilde{x})$, *then* f *is the identity.*

LEMMA 2. *In a compact Riemannian manifold* M, *any closed geodesic has diameter at least* $c(M)$, *and length at least* $2c(M)$. *Moreover, for* x, y *in* M *with* $d(x,y) < c(M)$, *there exists a unique minimizing geodesic joining* x *and* y.

LEMMA 3 (KLINGENBERG [4], TOPONOGOV [11]). *Let* M *be a compact simply connected Riemannian manifold with sectional curvature* K *satisfying* (*). *Then* $d(x, C(x)) \geqslant \pi/\sqrt{H}$ *for any* x *in* M.

Now the group of isometries $I(M)$ of a compact connected Riemannian manifold M is a compact Lie group. Hence if $I(M)$ is nontrivial, it contains a cyclic group. So we may assume that $I(M)$ contains at least a cyclic group of prime order p, for some p. Define the *modulo p equivariant diameter* $D_p(M)$ of M as the supremum of the number $\varepsilon > 0$ such that, for every nontrivial isometric action of cyclic group of order p on M, there exists an orbit of diameter at least ε. This implies that $D(M) \geqslant D_p(M)$ for some prime p. Thus, Theorems 1 and 2 follow immediately from the following theorem:

THEOREM 4. *Suppose* M *is a compact connected Riemannian manifold such that the group of isometries* $I(M)$ *contains a cyclic group* G *of prime order* p. *Then*

(1) $$D_p(M) \geqslant \begin{cases} c(M), & \text{if } p = 2, \\ (p-1)c(M)/p, & \text{if } p \neq 2. \end{cases}$$

In particular if the sectional curvature of M satisfies $(*)$, *then*

(2) $$D_p(M) \geqslant \begin{cases} \pi/|\pi_1(M)|\sqrt{H}, & \text{if } p = 2, \\ (p-1)\pi/p|\pi_1(M)|\sqrt{H}, & \text{if } p \neq 2. \end{cases}$$

PROOF. Let f be a fixed generator of G. Then f is an isometry of M. There are two possibilities:

(3) There exists $x_0 \in M$ such that $f(x_0)$ lies in the cut locus $C(x_0)$ of x_0.

(4) $f(x) \notin C(x)$ for all $x \in M$. We shall prove that the condition (4) implies

(5) There exists a nontrivial orbit Gx_0 (that is, not a fixed point) which is contained in a closed geodesic.

Define δ_f: $M \to R$ by $\delta_f(x) = d(x, f(x))$. Then the function δ_f^2: $M \to R$ is differentiable on M [10, p. 413]. Let x_0 be a critical point of δ_f^2 which is not a fixed point of f. Then $f(x_0) \neq x_0$. Since $f(x_0) \notin C(x_0)$ by (4), there exists a unique minimizing geodesic γ joining x_0 to $f(x_0)$. To prove (5), it suffices to show that the geodesic γ joins $f\gamma$ smoothly at fx_0. This follows from a result of Ozols [10, p. 414]. Let the closed geodesic which contains Gx_0 be γ'. Since f is an isometry, so the diameter of Gx_0 is equal to

$$\{(p-1)/2\} d(x_0, f(x_0)) = (p-1) \text{length of } \gamma'/2p$$

if p is odd. Hence

(6) $$\operatorname{diam} Gx_0 \geqslant \begin{cases} \text{diameter of } \gamma', & \text{if } p = 2, \\ (p-1) \text{ diameter of } \gamma'/p, & \text{if } p \neq 2, \end{cases}$$

where $\operatorname{diam} Gx_0$ denotes the diameter of the orbit Gx_0. It follows from (3), (6) and Lemma 2 that

$$D_p(M) \geqslant \begin{cases} c(M), & \text{if } p = 2, \\ (p-1)c(M)/p, & \text{if } p \neq 2. \end{cases}$$

To prove (2), it suffices to show that if the manifold M satisfies $(*)$ then $c(M) \geqslant \pi/|\pi_1(M)|\sqrt{H}$. This follows immediately from Lemma 3 if the manifold M is simply connected. In the case M is not simply connected, there is an n-fold covering p: $\tilde{M} \to M$ with $\tilde{M}$ simply connected, where $n = |\pi_1(M)|$. We can introduce a Riemannian metric on $\tilde{M}$ in such a way that p is an isometric immersion. Hence the sectional curvature $\tilde{K}$ of $\tilde{M}$ also satisfies $(*)$. Now $c(M)$ is given by the smaller one of the following two numbers:

(a) The minimum distance between two conjugate points in M which is greater than or equal to $\pi/\sqrt{H}$ [6, p. 78],

(b) Half of the minimum length of a closed geodesic in M.

Hence it remains to show that any closed geodesic in M has length greater than or equal to $2\pi/n\sqrt{H}$. Since $|\pi_1(M)| = n$, for any closed geodesic τ in M, there is a closed geodesic $\tilde{\tau}$ in $\tilde{M}$ with $p(\tilde{\tau}) = k\tau$ (that is, $\tilde{\tau}$ covers τ k-times) for some positive integer k which is a factor of n and $k \leqslant n$. It follows that if there is a closed geodesic in M of length less than $2\pi/n\sqrt{H}$, then there is a closed geodesic in $\tilde{M}$ of length less than $2\pi/\sqrt{H}$ which is impossible by Lemmas 2 and 3.

REMARK. Theorem 4 implies that

$$D_2(S^n) = \pi \quad \text{and} \quad D_p(S^n) = (p-1)c(S^n)/p = (p-1)\pi/p \quad \text{for } p \text{ odd.}$$

Hence the estimate in Theorem 4 is again best possible. This also proves the Dress conjecture which states that $D_p(S^n) \geqslant (p-1)\pi/p$ for p odd [3].

PROOF OF THEOREM 3. Let G be a nontrivial finite cyclic group of $I(M)$ with a fixed generator f. Suppose $fx \notin C(x)$ for all $x \in M$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with projection π. The isometry f has a unique lifting to an isometry $\tilde{f}$: $\tilde{M} \to \tilde{M}$ so that for every $\tilde{x} \in \tilde{M}$, $d(\tilde{x},\tilde{f}(\tilde{x})) = d(x,f(x))$ where $x = \pi(\tilde{x})$ by [10]. According to Lemma 1, f is the identity. This leads to a contradiction. This shows that there is at least an $x \in M$ such that $f(x) \in C(x)$, and so $\operatorname{diam} D(x) \geqslant d(x, C(x)) \geqslant c(M)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002