

ON THE NUMBER OF SEPARABLE LOCALLY CONVEX SPACES

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ABSTRACT. The number of distinct separable locally convex spaces is shown to be 2^{2^c} . The number of distinct separable and complete, or metrizable, or normed locally convex spaces is shown to be 2^c . There is no separable locally convex space that is quotient-universal for the class of separable locally convex spaces.

1. Introduction. Throughout this paper we consider only Hausdorff locally convex spaces over the fixed field K of either real or complex numbers. Two locally convex spaces E and F are called isomorphic, denoted by $E \cong F$, if there exists an isomorphism (i.e., a linear homeomorphism) mapping E onto F .

Let \mathcal{L} be a class of locally convex spaces. A space E belonging to \mathcal{L} is *subspace universal* (respectively, *quotient universal*) in \mathcal{L} if for each L in \mathcal{L} there exists a subspace F of E such that $L \cong F$ (respectively, $L \cong E/F$). In order to shorten our formulations, we denote by \mathcal{S} the class of all separable locally convex spaces, and by \mathcal{C} , \mathcal{M} , \mathcal{F} , \mathcal{N} , and \mathcal{B} the classes of all complete, metrizable, metrizable complete (Fréchet), normed, and normed complete (Banach) spaces in \mathcal{S} , respectively.

First, let us recall a few known facts. By the classical results of Banach and Mazur [1, p. 185], [2], the spaces $C = C[0, 1]$ and l_1 are, respectively, subspace universal in \mathcal{N} and quotient universal in \mathcal{B} . Mazur and Orlicz [9, p. 14] proved that the Fréchet space $C(R)$ of all continuous functions $f: R \rightarrow K$, endowed with the topology of uniform convergence on compact sets, or, more abstractly, the product C^{\aleph_0} of countably many copies of C , is subspace universal in \mathcal{M} . More recently, it was observed in [7] that the product C^c of c (continuum) copies of C serves as a subspace universal space in \mathcal{S} .

It is unknown whether the classes \mathcal{S} , \mathcal{C} , \mathcal{M} , \mathcal{N} , or \mathcal{F} possess a quotient universal space (cf. [11, p. 47]). We show that the answer is negative (Corollary 1) in the cases of \mathcal{S} , \mathcal{M} , and \mathcal{N} . This follows as an easy consequence of the main result:

THEOREM. \mathcal{S} contains a subset \mathcal{Y} consisting of 2^{2^c} mutually nonisomorphic spaces such that each member of \mathcal{S} is isomorphic to a member of \mathcal{Y} . The same assertion holds for \mathcal{C} , \mathcal{M} , and \mathcal{N} with the cardinality of \mathcal{Y} changed to 2^c .

In addition to yielding the number of "distinct" separable locally convex spaces, the method of proof of the Theorem also shows that every subspace of

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finite codimension of a separable locally convex space is separable (Corollary 3). An example shows this is not true for subspaces of countable codimension.

2. The main results.

PROOF OF THEOREM. Let $E = (E, t)$ be a locally convex space, $E' = (E, t')$ its topological dual, and E^* its algebraic dual. For each f in E^* , let t_f denote the weakest (locally convex) linear topology on E such that $t \leq t_f$ and f is continuous under t_f . Thus, if $E'_f = \text{sp}(E' \cup \{f\})$, then $t_f = \sup\{t, \sigma(E, E'_f)\}$, so that $E'_f = (E, t'_f)$.

By definition $t \leq t_f$, and the equality holds iff $f \in E'$. Note also that on $f^{-1}(0)$ the topologies t and t_f coincide, and since $f^{-1}(0)$ is a closed hyperplane in (E, t_f) , it follows that

$$(1) \quad (E, t_f) \cong f^{-1}(0) \oplus K,$$

where $f^{-1}(0)$ is endowed with the topology induced by t .

If $f, g \in E^*$, then $g \in E'_f$ iff $E'_g \subset E'_f$ and $E'_g \subset E'_f$ iff $t_g \leq t_f$. Since $E'_f = E'$ or E' is of codimension one in E'_f , $E'_g \subset E'_f$ holds iff $E'_g = E'$ or $E'_g = E'_f$. Hence $t_g \leq t_f$ iff $t_g = t$ or $t_g = t_f$. Therefore, if $g \notin E'$, then $t_g \leq t_f$ iff $g \in E'_f \setminus E'$. Writing $f \sim g$ when $t_f = t_g$, we define an equivalence relation on E^* ; let $[f]_{\sim} = \{g \in E^* : f \sim g\}$. Then $[f]_{\sim} = E'$ for each $f \in E'$, $[f]_{\sim} = E'_f \setminus E'$ for each $f \in E^* \setminus E'$, and $\text{card } [f]_{\sim} = \text{card } E'$ for each $f \in E^*$.

From now on let us assume that (E, t) is separable. Then, for each $f \in E^*$, the space (E, t_f) is separable, too. In fact, let D be a countable dense subset of E , and suppose D is not dense in (E, t_f) . Let L be the t_f -closed linear subspace of E spanned by D . By assumption, $L \neq E$. Hence there exists a nonzero g in E'_f such that $g(L) = \{0\}$. Since $g \notin E'$, we have $t_g = t_f$, and so by (1)

$$(E, t_f) = (E, t_g) \cong g^{-1}(0) \oplus K.$$

But $g^{-1}(0)$, in the topology induced by t , is evidently separable, hence (E, t_f) is separable as well.

Assume $\dim E = m$, where $m = c$ or $m = 2^c$. Then $\text{card } E^* = \dim E^* = 2^m$ and $\text{card } E' = c$ because E is separable. From each equivalence class $[f]_{\sim} \neq E'$ considered above choose an element f , and let F denote the set thus obtained. (Instead, we may take any Hamel basis B of E^*/E' and then define F to be a set of representatives chosen out of each member of B .) Evidently $\text{card } F = 2^m$, because $\text{card } [f]_{\sim} = c$ for each $f \in E^*$. Moreover, F has the property

$$(2) \quad f, g \in F \text{ and } f \neq g \text{ imply } t_f \text{ and } t_g \text{ are totally incomparable;}$$

that is, neither $t_f \leq t_g$ nor $t_g \leq t_f$ holds.

Now, given an f in F , we claim that (E, t_f) can be isomorphic to (E, t_g) for at most m members g of F . In order to see this, let D be a countable dense subset of (E, t_f) . If, for some $g \in F$, A is an isomorphism of (E, t_f) onto (E, t_g) , then A is completely determined by the values it assumes on D . Hence there are at most $m^{\aleph_0} = m$ distinct mappings A of E into E such that for some $g \in F$ (depending on A) A establishes the isomorphism of (E, t_f) and (E, t_g) . It is not excluded, however, that different g 's may correspond to the same A .

So suppose $g, h \in F$, $g \neq h$, and A is an isomorphism of (E, t_f) onto (E, t_g) and onto (E, t_h) . Then the identity AA^{-1} is an isomorphism of (E, t_g) onto (E, t_h) , and so $t_g = t_h$, which contradicts property (2) of F .

Now, for any $f, g \in F$, let $f \approx g$ mean that (E, t_f) and (E, t_g) are isomorphic. By what was just shown, each equivalence class $[f]_{\approx}$ contains at most m members of F , and since $\text{card } F = 2^m$, there are precisely 2^m distinct \approx -equivalence classes. Thus there exists a subset G of F with $\text{card } G = 2^m$ such that the separable locally convex spaces (E, t_g) , $g \in G$, are mutually nonisomorphic. These spaces are metrizable or normed if such is the space E .

Now specifying E to be C^c , we have $\dim C^c = 2^c$. The universality of E guarantees the existence of a maximal set \mathcal{G} of mutually nonisomorphic linear subspaces of E such that each member of \mathcal{S} is isomorphic to a member of \mathcal{G} . Clearly, we must have $\text{card } \mathcal{G} \leq 2^{2^c}$. On the other hand, each space (E, t_g) , $g \in G$, is isomorphic to a member of \mathcal{G} so that $\text{card } \mathcal{G} \geq 2^{2^c}$.

In the case of \mathfrak{N} and \mathfrak{U} , let E be C^{\aleph_0} and C , respectively, and note that $\dim C^{\aleph_0} = \dim C = c$. In either case, let \mathcal{G} be defined in a similar manner as before. Then $\text{card } \mathcal{G} \leq 2^c$ and a similar argument as before shows $\text{card } \mathcal{G} \geq 2^c$.

In the case of \mathcal{C} , let E be C^c and choose \mathcal{G} to be a maximal set of mutually nonisomorphic closed linear subspaces of E . Since each closed linear subspace of E is an intersection of closed hyperplanes and $\text{card } E' = c$, we have $\text{card } \mathcal{G} \leq 2^c$. On the other hand, for each nonempty subset P of $[1, \infty)$, let l_P denote the product $\prod_{p \in P} l_p$. Then l_P is complete and separable by the Hewitt-Marczewski-Pondiczery theorem (cf. [5, p. 77], [10]). If $P, Q \subset [1, \infty)$ and $P \neq Q$, then l_P and l_Q are not isomorphic. Indeed, suppose $P \not\subset Q$ and let $p \in P \setminus Q$. If A is an isomorphism of l_P onto l_Q , then its restriction to the factor space l_p is an embedding of l_p into l_Q . It follows from Proposition 3 of [3] that there exist $q_1, \dots, q_n \in Q$ such that $\prod_{i=1}^n l_{q_i}$ contains an isomorphic copy X of l_p . As l_p and l_{q_i} are well known to have no infinite-dimensional isomorphic subspaces, each coordinate projection of $\prod_{i=1}^n l_{q_i}$ onto X is strictly singular. However, the sum of strictly singular maps is strictly singular (cf. [6, p. 86]) so that the identity mapping on X is strictly singular. The contradiction shows that l_P and l_Q are not isomorphic. Since \mathcal{G} must contain an isomorphic copy of l_P for $\emptyset \neq P \subset [1, \infty)$, $\text{card } \mathcal{G} \geq 2^c$.

REMARK. It is well known that a set \mathcal{G} satisfying the statement of the Theorem can be chosen for the classes \mathfrak{F} and \mathfrak{B} with $\text{card } \mathcal{G} = c$.

COROLLARY 1. *There is no quotient-universal space in the classes \mathfrak{U} , \mathfrak{N} , and \mathfrak{S} .*

PROOF. Consider, for instance, the case of \mathfrak{S} . Suppose that a space $E \in \mathfrak{S}$ is quotient-universal in \mathfrak{S} . Then E admits at most 2^c distinct (up to isomorphism) Hausdorff quotients, because $\text{card } E' = c$ and each closed linear subspace of E is an intersection of a family of closed hyperplanes $f^{-1}(0)$, $f \in E'$. But there are 2^{2^c} mutually nonisomorphic separable locally convex spaces, yielding a contradiction. The proof in the cases of \mathfrak{N} and \mathfrak{U} is similar.

By a result of Dieudonné [4], every finite-codimensional subspace of a barrelled space is barrelled (and hence a Mackey space). The subspace-universal spaces E used in the proof of the Theorem are barrelled as products of barrelled spaces, and the spaces (E, t_f) are barrelled since they are

isomorphic to products of one-codimensional subspaces of E with K . Hence we have

COROLLARY 2. *There are precisely 2^{2^c} , 2^c and 2^c distinct barrelled (resp. Mackey) spaces in \mathfrak{S} , \mathfrak{M} , and \mathfrak{N} , respectively.*

Our next result is also an easy consequence of the proof of the Theorem (cf. [14]).

COROLLARY 3. *Every subspace of finite codimension of a separable locally convex space is separable.*

REMARK. In view of some recent results showing that certain properties of locally convex spaces are inherited by subspaces of countable codimension (see e.g. [12] and [13] for the property of being barrelled), one might ask whether, in Corollary 3, “finite” can be replaced by “countable”. We give a counterexample showing that this need not be the case. Note that there exists a separable locally convex space E with a closed nonseparable subspace F . An example of such a space was given in [8]. Let S be a countable subset of E whose closed linear span is all of E . Then $G = \text{sp}(F \cup S)$, in the relative topology, is a separable locally convex space whose countable-codimensional closed subspace F is not separable.

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