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ON LIE ALGEBRAS WITH PRIMITIVE ENVELOPES, SUPPLEMENTS

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ABSTRACT. Let L be a finite dimensional Lie algebra over a field k of characteristic zero, $U(L)$ its universal enveloping algebra and $Z(D(L))$ the center of the division ring of quotients of $U(L)$. A number of conditions on L are each shown to be equivalent with the primitive of $U(L)$. Also, a formula is given for the transcendency degree of $Z(D(L))$ over k.

1. **Introduction.** The aim of this paper is to establish a necessary and sufficient condition on a finite dimensional Lie algebra L over a field k in order that its universal enveloping algebra $U(L)$ is primitive. This settles a problem raised by Professor Jacobson [6, p. 23]. We may restrict ourselves to the case where k is of characteristic zero, since in characteristic $p \neq 0$, $U(L)$ is not primitive unless $L = 0$ [6, p. 255]. On the other hand, k is not assumed algebraically closed throughout the paper. Let $D(L)$ be the division ring of quotients of $U(L)$, $Z(D(L))$ its center. Let $G \subset \operatorname{Aut} L$ be the smallest algebraic group whose Lie algebra $L(G)$ contains $\operatorname{ad} L$ (i.e. $L(G)$ is the algebraic hull of $\operatorname{ad} L$ in $\operatorname{End} L$). G is called the adjoint algebraic group of L. For each linear functional $f \in L^*$ we define $L[f]$ to be the collection of all $x \in L$ such that $f(Ex) = 0$ for all $E \in L(G)$. $L[f]$ is a Lie subalgebra of L containing the center of L. One verifies that $L[f]$ is an ideal of $L(f)$, where $L(f)$ is the radical of the alternating bilinear form $(x,y) \to f([x,y])$ defined on L. Clearly $L[f] = L(f)$ if L is ad-algebraic. Furthermore, let $K(L)$ be the quotient field of the symmetric algebra $S(L)$, $K(L)^I$ the subfield of invariants of $K(L)$.

We can now state the main result.

THEOREM. *The following conditions are equivalent*:

(1) $L[f] = 0$ *for some* $f \in L^*$.

(2) G *admits an open dense orbit in* L^* *for its contragredient action on* L^*.

(3) $K(L)^I = k$.

(4) $Z(D(L)) = k$.

(5) $U(L)$ *is primitive*.

The proof uses some striking properties of the Dixmier-Duflo map [3, pp. 314–320] as well as some earlier results on the subject [7]. Finally, we shall verify that the number $t = \min_{f \in L^*} \dim L[f]$ is equal to the transcendency

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degree of $Z(D(L))$ over k. This follows directly from the isomorphism that exists between $K(L)^I$ and $Z(D(L))$ in the algebraically closed case [8].

2. It is understood that we consider the Zariski topology on L^*. We denote by $O(f)$ the orbit of $f \in L^*$ under the contragredient action of G on L^*. $O(f)$ is irreducible (since G is irreducible) and open in its closure [1, p. 98]. Following Dixmier we call $r = \min_{f \in L^*} \dim L(f)$ the index of L and $f \in L^*$ is called regular if $\dim L(f) = r$ [3, p. 51]. It is well known that the set L^*_{reg} of all regular linear functionals is an open dense G–stable subset of L^*. A similar property, concerning the Lie subalgebras $L[f]$, is obtained in the following.

Lemma 1. *For all $f \in L^*$ we have* $\dim L[f] + \dim O(f) = \dim L$. *Moreover, the collection Ω of all $f \in L^*$ such that* $\dim L[f] = t$ *is an open dense, G-stable subset of L^** ($t = \min_{f \in L^*} \dim L[f]$).

Proof. If $\{x_1, \ldots, x_n\}$ is a basis for L and $\{E_1, \ldots, E_m\}$ a basis for $L(G)$, then it is easily seen that

$$\dim L[f] = \dim L - \operatorname{rank}\,(f(E_i x_j)_{ij}).$$

On the other hand, the stabilizer $S(f)$ of $f \in L^*$ is a closed subgroup of G and

$$\dim O(f) = \dim G - \dim S(f) = \dim L(G) - \dim L(S(f)),$$

where $L(S(f))$, being the Lie algebra of $S(f)$, is the set of all $E \in L(G)$ such that $f \circ E = 0$. By considering the bilinear map $L(G) \times L \to k$ sending (E, x) into $f(Ex)$ we observe that

$$\dim L(S(f)) = \dim L(G) - \operatorname{rank}\,(f(E_i x_j)_{ij}).$$

Hence

$$\dim O(f) = \operatorname{rank}\,(f(E_i x_j)_{ij}) = \dim L - \dim L[f].$$

This takes care of the first part of the lemma.

In particular,

$$\max_{f \in L^*} \dim O(f) = \operatorname{rank}_{K(L)}\,((E_i x_j)_{ij}) = n - t.$$

Thus $\Omega = \{f \in L^* \mid \operatorname{rank}\,(f(E_i x_j)) = n - t\}$ and is therefore an open dense subset of L^*. Being the union of all orbits of maximum dimension, Ω is also G-stable.

The following is a result due to Gabriel [3, p. 159].

Theorem. *Let I be a two-sided ideal of $U(L)$. Then the following conditions are equivalent*:

(i) *I is absolutely primitive (i.e. $I \otimes k'$ is primitive in $U(L \otimes k')$ for every field extension k' of k).*

(ii) *There exists an algebraically closed extension k' of k such that $I \otimes k'$ is primitive in $U(L \otimes k')$.*

(iii) *I is primitive and the center of the ring of quotients of $U(L)/I$ reduces to k.*

We are now in a position to prove the main criterion.

THEOREM 1. *Let L be a Lie algebra over k. Then the following conditions are equivalent*:

(1) $L[f] = 0$ *for some* $f \in L^*$.

(2) G *admits an open dense orbit in* L^* *for its contragredient action on* L^*.

(3) $K(L)^I = k$.

(4) $Z(D(L)) = K$.

(5) $U(L)$ *is primitive.*

PROOF. The equivalence of (1), (3) and (4) has already been shown in [7], as well as the implication (5) $\Rightarrow$ (1). Let us now verify (1) $\Leftrightarrow$ (2). Assume $L[f] = 0$ for a suitable $f \in L^*$. Then Lemma 1 implies that $\dim O(f) = n = \dim L^*$. Consequently $O(f)$ is dense in L^*. It is even open in L^* since $O(f)$ is open in its closure. Conversely, if $O(f)$ is open and dense in L^* for some $f \in L^*$, then $\dim O(f) = n$ which forces $L[f] = 0$ (Lemma 1). Moreover, such an orbit is evidently unique (if $O(h)$, $h \in L^*$, is also open, then $O(f) \cap O(h) \neq \varnothing$ and thus $O(f) = O(h)$). Since $O(f)$ is the only orbit of maximum dimension, it follows that $O(f) = \Omega$. Therefore $\Omega \cap L^*_{\mathrm{reg}} \neq \varnothing$ implies that $\Omega \subset L^*_{\mathrm{reg}}$.

(1) $\Rightarrow$ (5). Let k' be the algebraic closure of k and put $L' = L \otimes k'$. Denote by Prim $(U(L'))$ the set of all primitive ideals of $U(L')$, endowed with the Jacobson topology [6, p. 203] and let J: $L'^*_{\mathrm{reg}} \to$ Prim $(U(L'))$ be the Dixmier-Duflo map which assigns to each regular functional $f \in L'^*$ a primitive ideal $J(f)$ of $U(L')$. J is known to be continuous [3, p. 317] and constant on the orbits lying in L'^*_{reg} (i.e. $J(g \cdot f) = J(f)$ for all $g \in G'$, G' being the algebraic adjoint of L') [3, p. 84], [8, p. 394]. Furthermore, if $Q \subset L'^*_{\mathrm{reg}}$ is dense in L'^* then $\bigcap_{f \in Q} J(f) = 0$ [3, p. 320].

In carrying out the proof of Lemma 1 we came across the formula

$$t = \min_{f \in L^*} \dim L[f] = \dim L - \operatorname{rank}_{K(L)} ((E_i x_j)_{ij})$$

whenever $\{E_1, \ldots, E_m\}$ is a basis for $L(G)$ and $\{x_1, \ldots, x_n\}$ a basis for L. Clearly this number t remains unchanged under extension of the base field k. So, if L satisfies (1) (i.e. $t = 0$) the same holds for L'. Then the foregoing observation shows that there exists an orbit $\Omega' \subset L'^*_{\mathrm{reg}}$ which is open dense in L'^*. Choose $h \in \Omega'$. Application of the properties of the map J mentioned above gives

$$J(h) = \bigcap_{f \in \Omega'} J(f) = 0.$$

Hence the ideal (0) is primitive in $U(L')$ and by Gabriel's theorem also in $U(L)$. This completes the proof.

REMARK. Because of this theorem, all examples of Lie algebra we have listed in [7] have primitive envelopes, even without the requirement that the base field k is algebraically closed.

Probably the most interesting class of Lie algebras satisfying the conditions of Theorem 1 is formed by the Lie algebras of index 0, partly because they include all ad-algebraic Lie algebras enjoying these conditions. If L is of index 0, it admits a linear functional $f \in L^*$ such that the alternating bilinear form on L sending (x,y) into $f([x,y])$ is nondegenerate, a situation reminiscent of Frobenius algebras in the associative case. In the study of these so called Frobenius Lie algebras, the Lie algebra of all $n \times n$ matrices with entries in k and with last row equal to zero seems to play a significant role. It is an ad-algebraic Frobenius Lie algebra satisfying the Gelfand-Kirillov conjecture [4], [7]. However, not all Frobenius Lie algebras are ad-algebraic (example b(iii) of [7, p. 497] is not even almost algebraic).

PROPOSITION. *Let L be a (finite dimensional) Lie algebra over k. If $f, f' \in L^*$ are such that $L[f] = 0 = L[f']$, then $f' = g \cdot f$ for some $g \in G$, G being the adjoint algebraic group of L. In particular, in a Frobenius Lie algebra any two regular linear functionals are conjugate by an element of the adjoint algebraic group.*

PROOF. We know from the proof of Theorem 1 that the set Ω of all $f \in L^*$ such that $L[f] = 0$ is an orbit under the action of G on L^*.

3. Next we want to establish a formula for the transcendency degree $\operatorname{tr}\deg_k(Z(D(L)))$ of the center $Z(D(L))$ over k. For this task we need to recall the following preliminary material.

Let s be the canonical linear isomorphism of $S(L)$ onto $U(L)$, which maps each product $y_1 \cdots y_q$, $y_i \in L$, into $(1/q!) \sum_p y_{p(1)} \cdots y_{p(q)}$ where p ranges over all permutations of $\{1,\dots,q\}$. Let $\{x_1,\dots,x_n\}$ be a basis of L and $\{E_1,\dots,E_m\}$ a basis for $L(G)$. Then $S(L) \cong k[X_1,\dots,X_n]$ is the direct sum of the subspaces S^q of homogeneous polynomials of degree q. On the other hand, let U_q, $q \geq 0$, be the family of subspaces of $U(L)$ which forms the usual increasing filtration of $U(L)$. The associated graded algebra is isomorphic to $S(L)$ by the Poincaré-Birkhoff-Witt theorem. The elements $u \in U_q \backslash U_{q-1}$ are said to be of degree q and $[u] = u \bmod U_{q-1}$ is called the leading term of u. All nonzero elements $u, v \in U(L)$ satisfy $[uv] = [u][v]$ and $\deg(uv) = \deg(u) + \deg(v)$. If $x = x_q + \cdots + x_0$, $x_q \neq 0$, is the decomposition of $x \in S(L)$ into homogeneous components $(x_i \in S^i)$ then we notice that $[s(x)] = x_q$. Every $E \in \operatorname{ad} L$ acts as a derivation in both $K(L)$ and $D(L)$, leaving stable the subspaces S^q and U_q, and commutes with s (i.e. $Es(x) = s(Ex)$ for all $x \in S(L)$).

In order to proceed we require the following lemmas.

LEMMA 2. *$K(L)^I$ is generated as a field by elements of the form $xy^{-1} \in K(L)^I$, $y \neq 0$, where x and y are homogeneous semi-invariants, i.e. $x \in S^i$, $y \in S^j$ for some $i, j \in \mathbf{N}$ and we can find a $\lambda \in (\operatorname{ad} L)^*$ such that $Ex = \lambda(E)x$, $Ey = \lambda(E)y$ for all $E \in \operatorname{ad} L$.*

PROOF. Let $u \in K(L)^I$. We may write $u = xy^{-1}$, $y \neq 0$, where $x, y \in S(L)$ are relatively prime. A standard argument shows that there is a $\lambda \in (\operatorname{ad} L)^*$ such that $Ex = \lambda(E)x$, $Ey = \lambda(E)y$ for all $E \in \operatorname{ad} L$. Let $x = x_p + \cdots + x_0$, $y = y_q + \cdots + y_0$ be the decomposition into homogeneous components $(x_i \in S^i, y_j \in S^j)$. Since each $E \in \operatorname{ad} L$ maps each S^i into itself we see that $Ex = Ex_p + \cdots + Ex_0$ is the corresponding decomposition of Ex. It follows that $Ex_i = \lambda(E)x_i$ and similarly $Ey_j = \lambda(E)y_j$ for all i, j and for all $E \in \operatorname{ad} L$. Finally,

$$u = xy^{-1} = \sum_i x_i y^{-1} = \sum_i \Big(\sum_j y_j x_i^{-1}\Big)^{-1}$$

(only those indices i are considered for which $x_i \neq 0$) where each $y_j x_i^{-1} \in K(L)^I$ satisfies the requirements of the lemma.

LEMMA 3. $\operatorname{tr}\deg_k (K(L)^I) \leqslant \operatorname{tr}\deg_k (Z(D(L)))$.

PROOF. The previous lemma guarantees that we can single out a transcendency basis for $K(L)^I$ of the form $x_1 y_1^{-1}, \ldots, x_t y_t^{-1}, y_i \neq 0$, where all $x_i, y_i \in S(L)$ are homogeneous semi-invariants. Put $u_i = s(x_i)$, $v_i = s(y_i)$ and $z_i = u_i v_i^{-1}$. We observe that for all $E \in \operatorname{ad} L$, $Eu_i = Es(x_i) = s(Ex_i) = \lambda(E)s(x_i) = \lambda(E)u_i$ and similarly $Ev_i = \lambda(E)v_i$. Consequently, $z_i \in Z(D(L))$ since

$$\begin{aligned} Ez_i &= E(u_i v_i^{-1}) = (Eu_i - u_i v_i^{-1} Ev_i)v_i^{-1} \\ &= (\lambda(E)u_i - u_i v_i^{-1}\lambda(E)v_i)v_i^{-1} = 0 \quad \text{for all } E \in \operatorname{ad} L. \end{aligned}$$

Clearly, it suffices to show that $z_1, \ldots, z_t$ are algebraically independent over k. Suppose we can find some $a_q \in k$, not all zero ($q = (q_1, \ldots, q_t)$) such that $\sum_q a_q z_1^{q_1} \cdots z_t^{q_t} = 0$. Let m_i be the largest exponent of z_i that appears nontrivially in this sum. Since u_i and v_i commute with each other we obtain, after multiplication with $v_1^{m_1} \cdots v_t^{m_t}$, that

$$\sum_q a_q u_1^{q_1} v_1^{m_1 - q_1} \cdots u_t^{q_t} v_t^{m_t - q_t} = 0.$$

Let m be the largest degree (as defined in the preliminaries) of all monomials appearing nontrivially in this sum and let Q be the set of all q's with $a_q \neq 0$ and corresponding with the monomials of degree m. Then it follows that

$$\sum_{q \in Q} a_q [u_1]^{q_1} [v_1]^{m_1 - q_1} \cdots [u_t]^{q_t} [v_t]^{m_t - q_t} = 0.$$

After dividing by $[v_1]^{m_1} \cdots [v_t]^{m_t}$ and taking into account that $[u_i] = [s(x_i)] = x_i$ and $[v_i] = [s(y_i)] = y_i$ we conclude that $\sum_{q \in Q} a_q (x_1 y_1^{-1})^{q_1} \cdots (x_t y_t^{-1})^{q_t} = 0$ which contradicts our original assumption.

LEMMA 4. *Let k' be an extension field of k and put $L' = L \otimes k'$. Then* $\operatorname{tr}\deg_k (Z(D(L))) \leqslant \operatorname{tr}\deg_{k'} (Z(D(L')))$.

PROOF. The identification of $U(L) \otimes k'$ with $U(L')$ results in an imbedding of $D(L) \otimes k'$ into $D(L')$ and thus $D(L)$ and k' are linearly disjoint in $D(L')$. Therefore we may consider $Z(D(L)) \otimes k' \subset Z(D(L'))$. Suppose $z_1, \ldots, z_p \in Z(D(L))$ are algebraically independent over k. This means that the monomials $z_1^{n_1} \cdots z_p^{n_p}$, $n_i \in \mathbf{N}$, are linearly independent over k. Hence $z_1^{n_1} \cdots z_p^{n_p} \otimes 1$, $n_i \in \mathbf{N}$, are linearly independent over k'. This implies that $z_1 \otimes 1, \ldots, z_p \otimes 1$ are algebraically independent over k'. The result then follows immediately.

THEOREM 2. *Let L be a Lie algebra over k, G its adjoint algebraic group acting on L^* and M the largest dimension of all orbits in L^*. Then $Z(D(L))$ and $(K(L))^I$ have the same transcendency degree over k, equal to the number* $t = \min_{f \in L^*} \dim L[f] = \dim L - M$.

PROOF. Let k' be the algebraic closure of k and put $L' = L \otimes k'$. Then the fields $Z(D(L'))$ and $K(L')^I$ are k'-isomorphic [8, p. 401]. This combined with the preceding lemmas yields

$$\operatorname{tr\,deg}_k (K(L)^I) \leqslant \operatorname{tr\,deg}_k (Z(D(L))) \leqslant \operatorname{tr\,deg}_{k'} (Z(D(L'))) = \operatorname{tr\,deg}_{k'} (K(L')^I).$$

On the other hand, $\operatorname{tr\,deg}_k (K(L)^I) = \operatorname{tr\,deg}_{k'} (K(L')^I)$. Indeed, we know that

$$\operatorname{tr\,deg}_k (K(L)^I) = \dim L - \operatorname{rank}_{K(L)} ((E_i x_j)_{ij}) \quad [7],$$

which we have seen (in the proof of Theorem 1) to be equal to $t = \min_{f \in L^*} \dim L[f] = \dim L - M$ and which does not change under field extension. Hence, we may conclude that

$$\operatorname{tr\,deg}_k (Z(D(L))) = \operatorname{tr\,deg}_k (K(L)^I) = t.$$

REMARK. In case L is ad-algebraic the formula we came across in the preceding discussion simplifies to

$$\operatorname{tr\,deg}_k (Z(D(L))) = \dim L - \operatorname{rank}_{K(L)} ([x_i, x_j])$$

which is now equal to the index of L.

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