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INVERSE AND INJECTIVITY OF PARALLEL RELATIONS INDUCED BY CELLULAR AUTOMATA

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ABSTRACT. Moore and Myhill showed that Garden-of-Eden theorem [2], [3]. A binary relation over the configurations is said to be "parallel" if it is induced by a cellular (tessellation) automaton. Richardson showed the equivalence between a parallel relation (a nondeterministic parallel map) with the quiescent state to be injective and its inverse to be parallel by the Garden-of-Eden theorem plus compactness of product topology [4]. This paper deals with the inverse and the injectivity when a cellular automaton is given that induces a parallel relation. We give an equivalent condition, concerning only the local map, for the inverse of a parallel relation to be parallel. Furthermore we show an equivalent condition, concerning only the local map, for the injectivity of a parallel map. Consequently, we note that these two conditions are represented by semirecursive predicates.

1. **Introduction.** A cellular automaton–also known as a tessellation structure–is a model of an array of uniformly connected identical finite automata arranged in a d-dimensional Euclidean space divided into square cells, where d is called the *dimension*. The *cellular automaton* is denoted by $M = (V, Z^d, X, f)$, where (i) V is the *state set* of each finite automaton. (ii) Z denoted the integers. (iii) X is a distinct n-tuple $(x_1, x_2, \dots, x_n)$ from Z^d, called the *neighbourhood index*, where n is a positive integer. We will always assume that $x_1 = \mathbf{0}$ ($\mathbf{0}$ denotes the p-tuple of 0's). X denotes the locations of the finite automata which are connected to each finite automaton. (iv) $f \subseteq V^n \times V$ is the state function of each finite automaton called the local relation. We often represent a binary relation $R \subseteq X \times Y$ by a nondeterministic mapping of a subset of X to 2^Y. A totally defined or a deterministic relation denotes a totally defined or a deterministic mapping, respectively.

A *configuration* is a mapping $Z^d \to V$, which is an assignment of states into the array. Now, the *parallel relation* R (over the configurations) *induced by* M is defined as follows: For configurations c and d,

$$(c, d) \in R \Leftrightarrow f(c(i + x_1), c(i + x_2), \dots, c(i + x_n)) \ni d(i)\ \forall i \in Z^d.$$

A binary relation over the configurations is said to be *parallel* if it is induced by some cellular automaton. A parallel relation R is called a parallel *map* if R is deterministic, that is, the local relation is deterministic.

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A cellular automaton $M = (V, Z^d, X, f)$ is *with the quiscent state* if there is a state v_q in V such that $f(v_q, \ldots, v_q) = \{v_q\}$. The parallel relation R induced by M is *with the quiescent state* if M is with the quiescent state, i.e., $c_q R d$ iff $d = c_q$ for the quiescent configuration c_q. A configuration c is *with finite support* provided that the set $\{i \in Z; c(i) \neq v_q\}$ is finite.

$\overline{X}(i)$ denotes the set $\{(i + x_1), (i + x_2), \ldots, (i + x_n)\}$ and $\overline{X}(A)$ denotes $\bigcup_{i \in A} \overline{X}(i)$. A *pattern* is a restriction of a configuration to a finite set. The *parallel relation* R_p *over the patterns* is defined by: For patterns p and q, $(p, q) \in R_p$ iff dom $p = \overline{X}$(dom q) and

$$(1) \qquad f(p(i + x_1), p(i + x_2), \ldots, p(i + x_n)) \ni q(i) \ \forall i \in \text{dom } q.$$

GARDEN-OF-EDEN THEOREM (Moore [2] and Myhill [3]). *A totally defined parallel map R with the quiescent state is surjective if and only if R is injective restricted to configurations with finite support.*

Richardson combined the theorem above and compactness of product topology [4], and gave the following theorem.

THEOREM A (Richardson [4]). *A totally defined parallel map R with the quiescent state is injective if and only if the inverse of R is a totally defined parallel map with the quiescent state.*

2. **Results.**

DEFINITION. Let R_p be a parallel relation over the patterns induced by $M = (V, Z^d, X, f)$. With respect to a finite set A ($\mathbf{0} \in A$) in Z^d, R_p is said to be *A-independent* if for any patterns p, p' and q such that dom $q = A$, $p R_p q$, and $p' R_p q$, $r R_p q$ for the pattern r such that dom r = dom p, $r(\mathbf{0}) = p'(\mathbf{0})$ and $r(i) = p(i)$ for $i \neq \mathbf{0}$.

A set A is said to be *sufficiently large with respect to* X if $\overline{X}(i) \cap \overline{X}(\mathbf{0}) = \emptyset$ or $\overline{X}(i) \cap (Z^d - \overline{X}(A)) = \emptyset$ for any i in Z^d.

LEMMA 1. *Let R^{-1} be the inverse of a totally defined parallel relation R induced by $M = (V, Z^d, X, f)$. If R^{-1} is a parallel relation induced by $M' = (V, Z^d, Y, g)$, then R_p is A-independent for some sufficiently large finite set A in Z^d.*

PROOF. Let A be a sufficiently large finite set in Z^d such that $\overline{Y}(\mathbf{0}) \subseteq A$ and $\mathbf{0} \in A$. Since R^{-1} is parallel, for patterns p, p' and q, if dom $q = A$, $p R_p q$ and $p' R_p q$, then $p(\mathbf{0})$ and $p'(\mathbf{0})$ are in

$$g(q(\mathbf{0} + y_1), q(\mathbf{0} + y_2), \ldots, q(\mathbf{0} + y_{n'})),$$

where $Y = (y_1, y_2, \ldots, y_{n'})$. Since f is totally defined, then there are patterns q_1 and q'_1 such that (i) $\overline{Y}(\overline{X}(A))$ = dom q_1 = dom q'_1, (ii) $q_1(i) = q'_1(i) = q(i)$ for i in A, and (iii) $q_1 (R^{-1})_p p$ and $q'_1 (R^{-1})_p p'$, where $(R^{-1})_p$ is the parallel relation over the patterns induced by M'. Thus from (1), $q_1 (R^{-1})_p r$ for the pattern r such that dom r = dom p, $r(\mathbf{0}) = p'(\mathbf{0})$ and $r(i) = p(i)$ for $i \neq \mathbf{0}$, since $\overline{Y}(\mathbf{0}) \subseteq A$. Accordingly, $r R_p q$, and therefore R_p is A-independent. Q.E.D.

In order to prove the converse of Lemma 1, we will show in advance the following

LEMMA 2. *Let us suppose that R is the parallel relation induced by $M = (V, Z^d, X, f)$. Let A be a finite set in Z^d which is sufficiently large with respect to X, and A' be such as $A' \supseteq A$. If R_p is A-independent, then R_p is A'-independent.*

PROOF. For any patterns p'_1, p'_2 and q' such that $\operatorname{dom} q' = A'$, if $p'_1 R_p q'$ and $p'_2 R_p q'$, then $p_1 R_p q$ and $p_2 R_p q$ for patterns p_1, p_2 and q such as $p_1 \subseteq p'_1, p_2 \subseteq p'_2, q \subseteq q'$, $\operatorname{dom} q = A$ and $\operatorname{dom} p_1 = \operatorname{dom} p_2 = \overline{X}(A)$. Accordingly $r R_p q$ for the pattern r such that $\operatorname{dom} r = \operatorname{dom} p$, $r(\mathbf{0}) = p_2(\mathbf{0})$ and $r(i) = p_1(i)$ for $i \neq \mathbf{0}$. Let r' be a pattern such as (i) $\operatorname{dom} r' = \overline{X}(A')$, (ii) $r'(\mathbf{0}) = p'_2(\mathbf{0})$, and (iii) $r'(i) = p'_1(i)$ for $i \neq \mathbf{0}$.

We will prove that $r' R_p q'$. Let r'' be the restriction of r' to $\overline{X}(i)$. For i such that $\overline{X}(i) \ni \mathbf{0}$, we obtain $r'' R_p q'(i)$ $(q'(i) = q(i))$, since $r''(i) = r(i)$ for $i \in \overline{X}(i)$, R is A-independent and A is sufficiently large. For i such as $\overline{X}(i) \not\ni \mathbf{0}$, it is obvious that $p''_1 R_p q'(i)$, where p''_1 is the restriction of p'_1 to $\overline{X}(i)$. Q.E.D.

The next theorem does not only show that the inverse of a parallel relation is parallel when R_p is A-independent, but also shows that we can explicitly give a cellular automaton that induces the inverse, using a cellular automaton defined below.

Let us suppose that $A = \{a_1, a_2, \ldots, a_m\}$ $(a_1 = \mathbf{0})$ is a finite set in Z^d and that $M = (V, Z^d, X, f)$ is a cellular automaton with $X = (x_1, x_2, \ldots, x_n)$. Let $(R_p)^{-1}$ be the inverse of the parallel relation R_p over the patterns induced by M. Let $Y = (y_1, y_2, \ldots, y_{n'})$ be such that

$$\overline{X}(A) \subseteq \bigcap_{1 \leqslant i \leqslant n} \overline{Y}(x_i) \tag{2}$$

and that n' is finite. Now, $M(A) = (V, Z^d, Y, g)$ denotes a cellular automaton defined as:

$$g(v_1, v_2, \ldots, v_n) \ni v, \tag{3}$$

if there are patterns p and q such that $p R_p q$, $p(\mathbf{0}) = v$ and $p(y_i) = v_i$ for any i $(1 \leqslant i \leqslant n')$ where $\operatorname{dom} p = \overline{X}(\overline{Y}(\mathbf{0}))$ and $\operatorname{dom} q = \overline{Y}(\mathbf{0})$.

THEOREM 1. *Let R be a parallel relation induced by $M = (V, Z^d, X, f)$ with the inverse R^{-1}. Let A be a sufficiently large finite set in Z^d with respect to X. If the parallel relation R_p over the patterns is A-independent, then a cellular automaton $M(A)$ defined in* (3) *induces R^{-1}.*

PROOF. Let A be such that $A = \{a_1, a_2, \ldots, a_m\}$. We can assume that $a_1 = \mathbf{0}$. Assume that $M(A) = (V, Z^d, Y, g)$, $Y = (y_1, y_2, \ldots, y_{n'})$ and S is the parallel relation induced by $M(A)$.

We will prove that $d R^{-1} c$ if and only if $d S c$ for configurations c and d. Let us assume that $d R^{-1} c$. Let p be the restriction of c to $\overline{X}(\overline{Y}(i))$ and q be the restriction of d to $\overline{Y}(i)$ with respect to any i in Z^d. Clearly $p R_p q$. Hence $p(i) \in g(q(i + y_1), q(i + y_2), \ldots, q(i + y_{n'}))$ from (3). Thus $d S c$.

On the other hand, let us assume that $d S c$. We must prove that $d(i) \in f(c(i + x_1), c(i + x_2), \ldots, c(i + x_n))$ for any i in Z^d, where $X = (x_1, x_2, \ldots, x_n)$. Fix $i \in Z^d$.

From (3) there are patterns $p_1, p_2, \ldots, p_n$ for each $1 \leqslant j \leqslant n$ such that

$$\begin{cases} \operatorname{dom} p_j = \overline{X}(\overline{Y}(i + x_j)), \\ p_j(i + x_j) = c(i + x_j), \\ p_j R_p q_j, \end{cases}$$

where $X = (x_1, x_2, \ldots, x_n)$ and q_j is the restriction of d to $\overline{Y}(i + x_j)$. Let q'_j $(1 \leqslant j \leqslant n)$ be such that $q'_j = q_1 \cap q_2 \cap \cdots \cap q_j$, where mappings q_1, $q_2, \ldots, q_j$ are considered as sets.

Let $r_1, r_2, \ldots, r_n$ be the patterns as defined below:

$$\begin{cases} r_1 = p_1, \\ \operatorname{dom} r_{j+1} = \operatorname{dom} r_j \cap \operatorname{dom} p_{j+1}, \\ r_{j+1}(x) = r_j(x) \qquad (x \neq i + x_{j+1}), \end{cases}$$
$$= p_{j+1}(x) \qquad (x = i + x_{j+1}),$$

where $1 \leqslant j \leqslant n$. It is clear that $r_1 R_p q'_1$. We will prove that if $r_j R_p q'_j$, then $r_{j+1} R_p q'_{j+1}$ for j $(1 \leqslant j < n)$.

Assume that $r_j R_p q'_j$. Let r'_{j+1} be the restriction of r_j to dom r_{j+1}. We have $r'_{j+1} R_p q'_{j+1}$, while A is sufficiently large and $\{i + x_{j+1} + a_1, i + x_{j+1} + a_2, \ldots, i + x_{j+1} + a_m\} \subseteq \operatorname{dom} r_{j+1}$ from (2). Thus, from Lemma 2, $r_{j+1} R_p q'_{j+1}$. Accordingly, $r_n R_p q'_n$. Since $q'_n(i) = d(i)$, the proof is completed. Q.E.D.

THEOREM 2. *The inverse of a totally defined parallel relation R induced by $M = (V, Z^d, X, f)$ is parallel if and only if the parallel relation R_p over the patterns is A-independent for some sufficiently large finite set A with respect to X.*

The next theorem deals with the injectivity of a parallel map. From Theorems A and 2

THEOREM 3. *A totally defined parallel map R with the quiescent state induced by $M = (V, Z^d, X, f)$ is injective if and only if*

(i) *the parallel map R_p over the patterns is A-independent for some sufficiently large finite set A with respect to X, and*

(ii) *the inverse R^{-1} is totally defined and with the quiescent state.*

We note that condition (i) in Theorem 3 is represented by a semirecursive predicate. While, R^{-1} in Theorem 3 is induced by $M(A)$ defined in (3), if R_p is A-independent. Then:

REMARK. The following are represented by semirecursive predicates:

(i) The inverse of the totally defined parallel relation induced by a given cellular automaton is parallel.

(ii) The totally defined parallel map with the quiescent state induced by a given cellular automaton is injective.

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