

REMARKS ON THE CLASSICAL BANACH OPERATOR IDEALS

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ABSTRACT. Sufficient conditions are given that the λ -tensor product of two operators be weakly compact.

Suppose W , X , Y , Z denote Banach spaces and $W \otimes_{\lambda} X$ denotes the topological tensor product of W and X under the least reasonable tensor cross-norm λ . If $T: W \rightarrow Y$ and $S: X \rightarrow Z$ are continuous linear operators then a continuous linear operator $W \otimes_{\lambda} X \rightarrow Y \otimes_{\lambda} Z$ is induced which may or may not share certain special properties enjoyed by T and S . The present note is concerned with the classical Banach operator ideal of weakly compact operators. While the λ -tensor product of weakly compact linear operators need not be weakly compact, if either of the operators in question is compact (Theorem 2) or if one of the operator's domains is a $C(K)$ - or an $L_1(\mu)$ -space then the λ -tensor product of weakly compact operators is again a weakly compact operator (Theorem 4).

In proving that the λ -tensor product of a weakly compact operator and a compact operator is again weakly compact no real use is made of weak compactness; indeed, Theorem 2 shows that for any classical injective Banach operator ideal an analogous statement holds.

A basic tool used in the proof of Theorem 4 is a recent, as yet unpublished result of W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński which states that every weakly compact linear operator between Banach spaces factors through a reflexive Banach space. We wish to thank Professors Davis, Figiel, Johnson, and Pełczyński for communicating their result. We also wish to thank Professor D. R. Lewis for conversations which led to the present proof of Theorem 4; this proof avoids the use of the representation theory of weakly compact operators on $C(K)$ - and $L_1(\mu)$ -spaces which we originally employed in proving Corollary 5.

Let I denote a classical Banach operator ideal; i.e., for each pair of Banach spaces X , Y , $I(X; Y)$ is a closed subspace of $L(X; Y)$ in the uniform norm topology containing the finite rank operators $F(X; Y)$ from X to Y and possessing the ideal property that if W , Z are Banach spaces and $T: W \rightarrow X$, $R: Y \rightarrow Z$ are bounded linear operators and $S \in I(X; Y)$, then $RST: W \rightarrow Z$ is a member of $I(W; Z)$. The reader is referred to [12] for a rather complete discussion of Banach operator ideals—classical and otherwise.

Among the classical examples of such structures one finds the classes of

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compact operators [1], weakly compact operators [1], strictly singular operators ([9], [10]), strictly cosingular operators ([7], [10]), completely continuous operators ($T \in L(X; Y)$ is said to be completely continuous if T maps weakly convergent sequences into norm convergent sequences), Dunford-Pettis operators (weakly compact completely continuous operators), weak Cauchy operators ($T \in L(X; Y)$ is said to be weak Cauchy whenever given a bounded sequence $(x_n) \subseteq X$, (Tx_n) has a weak Cauchy subsequence), and the unconditionally converging operators [8].

We say that a classical Banach operator ideal I is injective [12] whenever given any bounded linear operator $T: X \rightarrow Y$ and any $Z \supseteq Y$ (isomorphically) then $T \in I(X; Y)$ if and only if $T \in I(X; Z)$. Among the aforementioned examples for I , all but the strictly cosingular operator ideal are examples of injective classical Banach operator ideals.

That the strictly cosingular operator ideal is not injective is seen by considering any isomorphism T of c_0 into $C[0, 1]$. Clearly, such a T is not strictly cosingular, since Tc_0 is, by Sobczyk's theorem [7], complemented in $C[0, 1]$. Hence, the natural choices of φ, ψ allow one to complete the diagram

$$\begin{array}{ccc} c_0 & \xrightarrow{T} & C[0, 1] \\ & \searrow T = \varphi & \swarrow \psi \\ & Tc_0 & \end{array}$$

with φ, ψ both linear, continuous, onto maps. We now embed $C[0, 1]$ into $L_\infty[0, 1]$. But every operator $S: c_0 \rightarrow L_\infty[0, 1]$ is strictly cosingular. In fact, if E is any Banach space for which there exist epimorphisms $\varphi: c_0 \rightarrow E$, $\psi: L_\infty[0, 1] \rightarrow E$, then as a quotient of c_0 , E is separable. However, by one of Grothendieck's theorems ([2, pp. 169–170]) separable quotients of $L_\infty[0, 1]$ are reflexive. Thus E is a reflexive quotient of c_0 , i.e., φ is weakly compact; but weakly compact linear operators on c_0 are compact. Thus E is a compact quotient of c_0 , i.e., $\dim E < \infty$.

REMARK. Using the results of [11] and [13] one can show that if Ω_1 is a dispersed compact Hausdorff space and Ω_2 is an F -space then every simultaneous Banach space quotient of $C(\Omega_1)$ and $C(\Omega_2)$ is finite dimensional. Hence every $T: C(\Omega_1) \rightarrow C(\Omega_2)$ or $T: C(\Omega_2) \rightarrow C(\Omega_1)$ is strictly cosingular. Whether or not this is so for any Ω_2 such that $C(\Omega_2)$ is a Grothendieck space is not known.

THEOREM 1. Let I be a classical Banach operator ideal. Denote, for any pair of Banach spaces X, Y , by $J_I^\lambda(X; Y)$ the class of all bounded, linear operators $T: X \rightarrow Y$ such that if X_1, Y_1 are Banach spaces and $S \in I(X_1; Y_1)$ then $T \otimes_\lambda S \in I(X \otimes_\lambda X_1; Y \otimes_\lambda Y_1)$ (here λ denotes the least reasonable norm of [4]). Then J_I^λ is a classical Banach operator ideal contained in I .

PROOF. Let $T, S \in J_I^\lambda(X; Y)$ and suppose $R: X_1 \rightarrow Y_1$ is in $I(X_1; Y_1)$. Then $T \otimes_\lambda R, S \otimes_\lambda R$ are in $I(X \otimes_\lambda X_1; Y \otimes_\lambda Y_1)$ so

$$(T + S) \otimes_{\lambda} R = (T \otimes_{\lambda} R) + (S \otimes_{\lambda} R)$$

is also in $I(X \otimes_{\lambda} X_1; Y \otimes_{\lambda} Y_1)$ which yields $T + S \in J_f^{\lambda}(X; Y)$.

Now let W, Z be Banach spaces and let $T \in L(W, X)$, $S \in J_f^{\lambda}(X, Y)$ and $R \in L(Y, Z)$. Suppose $U: X_1 \rightarrow Y_1$ is a member of $I(X_1; Y_1)$. Then the diagram

$$\begin{array}{ccc} W \otimes_{\lambda} X_1 & \xrightarrow{RST \otimes_{\lambda} U} & Z \otimes_{\lambda} Y_1 \\ T \otimes_{\lambda} \text{id}_{X_1} \downarrow & & \uparrow R \otimes_{\lambda} \text{id}_{Y_1} \\ X \otimes_{\lambda} X_1 & \xrightarrow{S \otimes_{\lambda} U} & Y \otimes_{\lambda} Y_1 \end{array}$$

commutes showing that $RST \otimes_{\lambda} U$ is in $I(W \otimes_{\lambda} X_1; Z \otimes_{\lambda} Y_1)$ and hence $RST \in J_f^{\lambda}(W; Z)$.

Thus J_f^{λ} is an ideal.

We next show that J_f^{λ} contains all the finite dimensional continuous operators. Indeed, let $T: X \rightarrow Y$ be a one dimensional linear continuous operator between the Banach spaces X, Y . Then T is of the form $Tx = f(x)y$ for some $f \in X'$ and $y \in Y$. Now, if X_1, Y_1 are Banach spaces and $S \in I(X_1; Y_1)$, the following diagram commutes:

$$\begin{array}{ccc} X \otimes_{\lambda} X_1 & \xrightarrow{T \otimes_{\lambda} S} & Y \otimes_{\lambda} Y_1 \\ \pi_f \downarrow & & \uparrow y \otimes \\ X_1 & \xrightarrow{S} & Y_1 \end{array}$$

where $\pi_f(\sum_{i=1}^n x_i \otimes x_i^{(1)}) = \sum_{i=1}^n f(x_i)x_i^{(1)}$ and $y \otimes(y^1) = y \otimes y^1$. Thus $T \otimes_{\lambda} S \in I(X \otimes_{\lambda} X_1; Y \otimes_{\lambda} Y_1)$ and hence $T \in J_f^{\lambda}(X; Y)$. Linearity of J_f^{λ} yields the containment of all the finite rank continuous operators in J_f^{λ} .

Finally, the inequalities

$$\|T \otimes_{\lambda} S - T_n \otimes_{\lambda} S\| = \|(T - T_n) \otimes_{\lambda} S\| \leq \|T - T_n\| \cdot \|S\|$$

show that $J_f^{\lambda}(X; Y)$ is a closed subspace of $L(X; Y)$ hence is a classical Banach operator ideal.

That $J_f^{\lambda} \subseteq I$ follows from the commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ 1_{\otimes} \downarrow & & \uparrow \pi_1 \\ X \otimes_{\lambda} S & \xrightarrow{T \otimes_{\lambda} \text{id } S} & Y \otimes_{\lambda} S \end{array} \quad (S = \text{scalars})$$

where $1_{\otimes}(x) = x \otimes 1$ and $\pi_1(\sum_{i=1}^n y_i \otimes e_i) = \sum_{i=1}^n e_i y_i$.

Our next result contains as a very special case a theorem of J. R. Holub [6] in the case where I is the compact operator ideal.

THEOREM 2. *Suppose that I is an injective classical Banach operator ideal. Then $J_I^\lambda(X; Y)$ is an injective classical Banach operator ideal; consequently, J_I^λ contains the compact operator ideal.*

PROOF. The first assertion follows from the fact that λ is an injective tensorial norm, i.e., if $Y \subseteq Z$ and X is given then $X \otimes_\lambda Y \subseteq X \otimes_\lambda Z$ and $Y \otimes_\lambda X \subseteq Z \otimes_\lambda X$. This combined with the injectivity of I yields the easy proof of the injectivity of J_I^λ .

The second assertion is a consequence of the following fact: every injective classical Banach operator ideal contains the compact operator ideal. Indeed, by the definition of operator ideal every ideal I contains the class of finite rank operators. If I is injective and X, Y are Banach spaces then Y is isometrically isomorphic to a subspace of some space $C(\Omega)$ of continuous real valued functions on a compact Hausdorff space Ω . So, given $T: X \rightarrow Y$ compact, $T: X \rightarrow C(\Omega)$ is compact and hence the limit of finite rank operators (since by [3, pp. 180–186], $C(\Omega)$ possesses the metric approximation property). Therefore, $T \in I(X; C(\Omega))$. By injectivity of I , $T \in I(X; Y)$.

COROLLARY 3. *The λ -tensor product of a compact operator with a*

- (i) *(weakly) compact operator is (weakly) compact;*
- (ii) *strictly singular operator is strictly singular;*
- (iii) *unconditionally converging operator is unconditionally converging;*
- (iv) *completely continuous operator is completely continuous;*
- (v) *Dunford-Pettis operator is a Dunford-Pettis operator;*
- (vi) *weak Cauchy operator is a weak Cauchy operator.*

EXAMPLES. (i) The λ -tensor product of weakly compact operators need not be weakly compact; indeed, the identity operator $\text{id}: l_2 \rightarrow l_2$ is weakly compact yet

$$\text{id} \otimes_\lambda \text{id} = \text{id}_{l_2 \otimes_\lambda l_2}: l_2 \otimes_\lambda l_2 \rightarrow l_2 \otimes_\lambda l_2$$

is not weakly compact by the well-known nonreflexivity of $l_2 \otimes_\lambda l_2$.

(ii) The λ -tensor product of strictly singular operators need not be strictly singular; similarly, the λ -tensor product of unconditionally converging (strictly cosingular). The same example suffices in each case, namely, let i denote the canonical inclusion of l_2 into c_0 . Then $i \otimes_\lambda i: l_2 \otimes_\lambda l_2 \rightarrow c_0 \otimes_\lambda c_0$ fixes the diagonal $e_j \otimes e_j$ (e_j denotes the j th unit vector) and hence is an isomorphism of the closed linear span of $\{e_j \otimes e_j: j = 1, 2, \dots\}$ in $l_2 \otimes_\lambda l_2$ with the closed linear span of $\{e_j \otimes e_j: j = 1, 2, \dots\}$ in $c_0 \otimes_\lambda c_0$. In each case, by Theorem 5.5 of [5], this space is a complemented subspace isomorphic to c_0 . The assertions about $i \otimes_\lambda i$ now follow from the fact that $i: l_2 \rightarrow c_0$ is a strictly singular, strictly cosingular, unconditionally converging operator; while clearly $i \otimes_\lambda i: l_2 \otimes_\lambda l_2 \rightarrow c_0 \otimes_\lambda c_0$ is none of these.

THEOREM 4. *Let W be a Banach space whose dual space possesses the approximation property and the Dunford-Pettis property. Let $T: W \rightarrow Y$ and*

$S: X \rightarrow Z$ be weakly compact linear operators. Then $T \otimes_\lambda S: W \otimes_\lambda X \rightarrow Y \otimes_\lambda Z$ is a weakly compact linear operator.

PROOF. We start by noting a consequence of the factorization result of Davis-Figiel-Johnson-Pełczyński; we may assume $X = Z$ is reflexive and S is the identity on X . Indeed, if R is a reflexive Banach space such that for some $A \in L(X; R)$ and $B \in L(R; Z)$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{S} & Z \\ & \searrow A & \nearrow B \\ & R & \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} W \otimes_\lambda X & \xrightarrow{T \otimes_\lambda S} & Y \otimes_\lambda Z \\ \text{id}_W \otimes_\lambda A \downarrow & & \uparrow \text{id}_Y \otimes_\lambda B \\ W \otimes_\lambda R & \xrightarrow{T \otimes_\lambda \text{id}_R} & Y \otimes_\lambda R \end{array}$$

also commutes. Hence the weak compactness of $T \otimes_\lambda \text{id}_R$ implies that of $T \otimes_\lambda S$.

So we are reduced to the question of tensoring a weakly compact operator on a space W whose dual possesses the approximation and Dunford-Pettis properties with the (weakly compact) identity operator on a reflexive Banach space to yield (relative to the λ cross-norm) a weakly compact operator.

Suppose we consider any Banach space U whose dual possesses the approximation property and any reflexive Banach space V . Then the dual of $(U \otimes_\lambda V)$ is precisely the space of integral bilinear functionals ($B^\gamma(U, V)$ in the notation of [4]) on $U \times V$. Identifying $B^\gamma(U, V)$ with a space of operators we have $(U \otimes_\lambda V)' = \text{integral operators from } U \text{ to } V'$. By the Dunford-Pettis-Phillips theorem ([3, p. 134]), every integral operator into the reflexive space V' is nuclear; by the approximation assumption on U' , the nuclear operators from U to V' coincide with $U' \otimes_\gamma V'$. Thus under the above assumptions on U and V we have

$$(U \otimes_\lambda V)' = (U' \otimes_\gamma V').$$

Of course, by the universal mapping principle for γ we have $(U' \otimes_\gamma V')'$ identifiable with $L(U'; V'') = L(U'; V)$.

Now let T, S, W, X, Y, Z be as in the statement of the theorem. By the injectivity of the weakly compact operator ideal in order to show $T \otimes_\lambda S: W \otimes_\lambda X \rightarrow Y \otimes_\lambda Z$ is weakly compact it suffices to show $T \otimes_\lambda S$ is weakly compact into any super space of $Y \otimes_\lambda Z$. In particular, embedding if necessary Y into a $C(\Omega)$ space isometrically and using the injectivity of the λ

cross-norm we may and do assume that Y and all of its duals possess the (metric) approximation property.

Thus in light of the above paragraphs, we are in the following situation: W' and Y' both possess the approximation property and W' also possesses the Dunford-Pettis property. $T: W \rightarrow Y$ is weakly compact and R is a reflexive Banach space. We wish to show that

$$T \otimes_{\lambda} \text{id}_R: W \otimes_{\lambda} R \rightarrow Y \otimes_{\lambda} R$$

is weakly compact, i.e., that ([1, VI, 4.2])

$$(T \otimes_{\lambda} \text{id}_R)''(L(W'; R)) \subseteq Y \otimes_{\lambda} R$$

where by the third paragraph we have $L(W'; R) = (W \otimes_{\lambda} R)''$ and $L(Y'; R) = (Y \otimes_{\lambda} R)''$.

The action of $(T \otimes_{\lambda} \text{id}_R)''$ on $\varphi \in L(W'; R)$ is readily checked to be given by

$$(T \otimes_{\lambda} \text{id}_R)''(\varphi) = \varphi \cdot T' \in L(Y'; R).$$

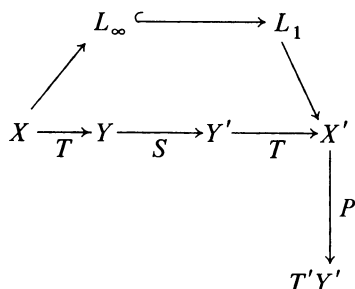
To show that $\varphi \cdot T' \in Y \otimes_{\lambda} R$ we must show (by the approximation considerations for Y) that $\varphi \cdot T'$ is a weak-star to weak, continuous compact linear operator from Y' to R . Weak star-weak continuity is immediate from T' 's weak star-weak continuity (VI, 4.7 of [1]) and the weak continuity of φ (V, 3.15 of [1]). The compactness of $\varphi \cdot T'$ follows from the fact that $T': Y' \rightarrow W'$ is weakly compact (VI, 4.18 of [1]) hence T' (Y' 's unit ball) is relatively weakly compact in W' which, since W' possesses the Dunford-Pettis property and $\varphi: W' \rightarrow R$ is necessarily weakly compact (and therefore maps relatively weakly compact subsets of W' into norm compact subsets of R), yields $\varphi \cdot T'$ (Y' unit ball) relatively compact in R .

COROLLARY 5. *If W is either a $C(K)$ -space (K a compact Hausdorff space) or an $L_1(\mu)$ -space then $T \otimes_{\lambda} S: W \otimes_{\lambda} X \rightarrow Y \otimes_{\lambda} Z$ is weakly compact whenever T, S are.*

We thank the referee and Professor T. H. E. Skulker for providing the following lemma and example in answer to our questions in the original version of the paper.

LEMMA. *Suppose $T: X \rightarrow Y$ is onto and $T'Y'$ is complemented in X' . Then $T \otimes_{\lambda} T: X \otimes_{\lambda} X \rightarrow Y \otimes_{\lambda} Y$ is onto.*

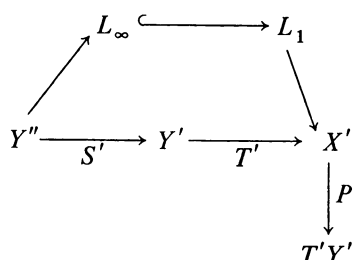
PROOF. We prove that $T \otimes_{\lambda} T$ is onto by showing that $(T \otimes_{\lambda} T)': L_{\gamma}(Y, Y') \rightarrow L_{\gamma}(X, X')$ is an isomorphism, where $L_{\gamma}(X, X')$ denotes the integral operators from X into X' . Let $P: X' \rightarrow T'Y'$ be a projection and recall that for $S \in L_{\gamma}(Y, Y')$, $(T \otimes_{\lambda} T)'(S) = T'ST$. Since for $S \in L_{\gamma}(Y, Y')$, both ST and $T'ST$ are integral operators, we have the following diagram:



which shows that

$$(1) \quad \|ST\|_\gamma = \|PT'ST\|_\gamma \leq \|P\| \cdot \|T'ST\|_\gamma.$$

The dual diagram



yields that

$$(2) \quad \|S\|_\gamma = \|S'\|_\gamma = \|PT'S'\|_\gamma \leq \|P\| \cdot \|T'S'\|_\gamma = \|ST\|_\gamma.$$

Thus if $(T \otimes_\lambda T)'S = T'ST = 0$ we have by (1) and (2) that $S = 0$.

Using this result we give an example to show that the λ -tensor product of Dunford-Pettis type operators is not an operator of Dunford-Pettis type.

Let $X = (\sum_n l_2^n)_l$ and $T: X \rightarrow l_2$ be the map defined by $Te_i^n = e_i$, where $(e_i)_{i=1}^n$ is the unit vector basis for l_2^n and (e_i) is the unit vector basis for l_2 . To show that T is onto and $T'l_2$ is norm one complemented in X' , we show that T admits local selections and use Stegall's local selection lemma ([14, Lemma 1]). If G is any finite dimensional Banach space and $S: G \rightarrow l_2$, letting f_1, f_2, \dots, f_n ($\dim G = n$) denote an orthonormal basis of SG , $[(f_j)_{j=1}^n]$ is isometrically isomorphic to l_2^n . Letting $\hat{S} = S$ with $f_i = e_i$, T admits a local selection with $\lambda = 1$. It is clear that X has the Dunford-Pettis property and T is weakly compact. $T \otimes_\lambda T: X \otimes_\lambda X \rightarrow l_2 \otimes_\lambda l_2$ is onto (by previous lemma) and hence not weakly compact by the nonreflexivity of $l_2 \otimes_\lambda l_2$.

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