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ON BOUNDED po-SEMIGROUPS

ZAHAVA SHMUELY

ABSTRACT. The bounded po-semigroup S is investigated by studying its increasing elements u ($\leqslant u^2$) and decreasing elements v ($\geqslant v^2$). In particular, in S, 01 ($= 0^n 1^m$), 10 ($= 1^n 0^m$), 010 and 101 are all idempotents and $010 = 01 \wedge_E 10$, $101 = 10 \vee_E 01$, E the set of idempotents of S ordered as a subset of S. In S, $0a1 = 01$ and $1a0 = 10$ holds for each $a \in S$. Consequently, S has a zero element z iff $01 = 10$ and in that case $z = 01$. S cannot be cancellative unless it is trivial. $J_0 = S10S \subseteq S$ is the kernel of S and consists of all (idempotents) $a \in S$ satisfying $aSa = a$. Thus when S is a (zero) simple bounded po-semigroup then $aSa = \{a, z\}$ and either $a^2 = a$ or $a^2 = z$ for each $a \in S$. When $S = X^X$, the po-semigroup of isotone maps f on the bounded poset X, then J_0 consists of all constant maps on X, hence $J_0 \simeq X$. The following generalization of Tarski's fixed point theorem is obtained: Let S be a complete (lattice and a) po-semigroup and let $s \in S$ be given. Then the set E_s (J_s) of all elements $x_0 \in E$ ($\in J_0$ resp.) satisfying $sx_0 = x_0 s = x_0$ is a nonempty complete lattice when ordered as a subset of S.

1. Let S denote a partially ordered (po) semigroup [1], [3]. Thus S is a semigroup endowed with a partial order $\leqslant$, such that $a \leqslant b \in S$ implies $ac \leqslant bc$ and $ca \leqslant cb$ for each $c \in S$. S is *bounded* if S contains universal bounds 0, 1 such that $0 \leqslant s \leqslant 1$ for each $s \in S$. If S is a po-semigroup which is a complete lattice with respect to $\leqslant$ we say that S is a *complete* po-semigroup. An element $z(i)$ satisfying $zs = sz = z$ ($is = si = s$ resp.) for each $s \in S$ is a *zero* (*identity* resp.) element of the semigroup S [2]. In general, 0, 1 must not be interchanged with z and i. Thus, e.g., let $S = X^X$ [1], be the po-semigroup of isotone maps f: $X \to X$, X a bounded poset. The semigroup operation in X^X is function composition and the order is the pointwise partial order. In X^X we clearly have $01 = 0$, $10 = 1$.

In this paper we study algebraic properties of a po-semigroup S. Two classes of elements, more general than the classes of positive and negative elements [3, p. 154], usually studied in po-semigroups, are introduced and shown to be of special significance. Thus $u \in S$ is *increasing* if $u \leqslant u^2$, and $v \in S$ is *decreasing* if $v^2 \leqslant v$. Obviously, e is both increasing and decreasing iff e is *idempotent*, namely $e = e^2$. Let $U, V \subseteq S$ denote the sets of increasing (decreasing resp.) elements of S. U, V and $E = U \cap V$ are ordered as subsets of S. Increasing and decreasing elements of X^X are treated in [6] and shown to form natural extensions of closure and anticlosure operators. As noted in [6, §7], it is the combined increasing-decreasing character of the identity i

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which is mainly responsible to the peculiar properties of Galois connections [5], [1] between two posets.

The existence in a po-semigroup S of elements u, v such that $u \in U$, $v \in V$ and $u \leqslant v$ (which is the case, e.g., for 0, 1 when S is bounded since $0 \leqslant 0^2 \leqslant 1^2 \leqslant 1$) gives rise to results of basic importance. Thus uv, vu, uvu and vuv are idempotents with $uvu = uv \wedge_E uv$, $vuv = vu \vee_E uv$. The structure of the subsemigroup S^* generated by $\{u, v\}$ is determined. For a bounded po-semigroup S we have: S contains a zero z iff $01 = 10$ holds, and in that case $z = 01$. S cannot be cancellative unless it is trivial. S has a nonempty *kernel* $J_0 = S10S$, which is the minimal ideal of S and $x \in J_0$ iff $x = x^2 = xSx$. When $S = X^X$, J_0 consists of all constant maps, hence $J_0 \approx X$. A characterization of (zero) simple bounded po-semigroups follows. Thus, e.g., when S is zero-simple, either $z = 1$ or $z = 0$, and for each $x \in S$ either $x^2 = x$ or $x^2 = z$ and $xSx = \{x, z\}$ for $x \neq z$. Consequently a (zero) simple bounded po-semigroup is completely (zero) simple.

It is interesting to note that an analogue of Tarski's fixed point theorem [7], [1, p. 115], holds in an arbitrary complete po-semigroup S. Thus for each $s \in S$ the set of idempotent elements x_0 $(\in E) \in J_0$ satisfying $x_0 s = s x_0 = x_0$ is a nonempty complete lattice when ordered as a subset of S. When applied to $S = X^X$ this is precisely Tarski's theorem.

2. We deal here with some basic algebraic properties of po-semigroups. At first we have:

LEMMA 1.2. *Let $v^2 \leqslant v \in S$ where S is a po-semigroup, and let $x \leqslant v$. Then*: (i) $xv \leqslant v$, $vx \leqslant v$; (ii) *both vx and xv are decreasing elements.*

PROOF. (i) follows by $xv \leqslant v^2 \leqslant v$ and $vx \leqslant v^2 \leqslant v$. To prove (ii) note that $(xv)^2 \leqslant xv^3 \leqslant xv^2 \leqslant xv$. $(vx)^2 \leqslant vx$ follows similarly.

Dually we get:

LEMMA 1*.2. *Let $u \leqslant u^2 \in S$ where S is a po-semigroup, and let $y \geqslant u$. Then*: (i) $yu \geqslant u$, $uy \geqslant u$; (ii) *both uy and yu are increasing elements.*

COROLLARY 1.2. *If $v \in S$ is decreasing, then $(v]$, the principal (order) ideal generated by v is a convex po-subsemigroup of S. If $u \in S$ is increasing, then $[u)$, the principal dual (order) ideal generated by u is a convex po-subsemigroup of S.*

If in a po-semigroup S a pair (u, v) of elements is given such that u in increasing, v is decreasing and $u \leqslant v$ then the chain of inclusions

$$(1) \qquad u \leqslant u^2 \leqslant \cdots \leqslant u^n \leqslant \cdots \leqslant v^n \leqslant \cdots \leqslant v^2 \leqslant v$$

is obtained. Note that in a po-group the existence of such a pair (u, v) is impossible unless $u = v = i$. The following theorem plays a central role in what follows.

THEOREM 1.2. *Let S be a po-semigroup and let $u, v \in S$ be given such that u is increasing, v is decreasing and $u \leqslant v$. Then*: (i) *for each $x \in [u, v]$ we have $uxv = uv$, $vxu = vu$*; (ii) $u^n v^m = uv$, $v^n u^m = vu$ *for $n, m \geqslant 1$*; (iii) *the elements uv, vu, uvu, and vuv are idempotents*; (iv) $uvu = uv \wedge_E vu = uv \wedge_U vu$, *and* $vuv = vu \vee_E uv = vu \vee_V uv$.

PROOF. If $u \leqslant x \leqslant v$ then $uv \leqslant u^2v \leqslant uxv \leqslant uv^2 \leqslant uv$ and $uxv = uv$. Similarly, we get $vxu = vu$ and (i) follows. (ii) is a special case of (i) since $u^kv^l, v^ku^l \in [u,v]$ for $k, l > 0$. To prove (iii) note that by Lemmas 1.2(ii), 1*.2(ii) the elements uv, vu are both increasing and decreasing, hence are idempotent. Since $u \leqslant u^2 \leqslant (uv)^2 = uv, vu = (vu)^2 \leqslant v^2 \leqslant v$, the last mentioned result applied again yields that both uvu and vuv are idempotent. To prove (iv) note that $uvu \leqslant uv, vu$. Now if $t \leqslant t^2 \in S$ satisfies $t \leqslant uv$, $t \leqslant vu$ then obviously $t \leqslant t^2 \leqslant uvvu = uvu$ by (ii) and $uvu = uv \wedge_E vu = uv \wedge_U vu$ follows. The second part of (iv) follows similarly.

COROLLARY 2.2. *If a po-semigroup S contains elements u, v such that $u \leqslant u^2 \leqslant v^2 \leqslant v$ then $E \neq \varnothing$.*

Note that under the conditions of Theorem 1.2 it acutally follows that if $u \leqslant x \leqslant y \leqslant v$ and if u_1 denotes either ux or xu while v_1 denotes either vy or yv then both u_1v_1 and v_1u_1 are idempotents with $u_1v_1u_1 = u_1v_1 \wedge_E v_1u_1$, $v_1u_1v_1 = v_1u_1 \vee_E u_1v_1$.

COROLLARY 3.2. *Let S be a po-semigroup and let $u, v \in S$ satisfy $u \leqslant u^2 \leqslant v^2 \leqslant v$. Then $\hat{S}$, the subsemigroup generated by $\{u,v\}$ is an order–and semigroup–epimorphic image of S^*, the "free" po-semigroup generated by u and v where*

$$S^* = \{u^n\}_{n=1,2,\dots} \cup \{v^n\}_{n=1,2,\dots} \cup \{uvu, uv, vu, vuv\}. \tag{2}$$

Notice that the last four elements in (2) form an idempotent po-subsemigroup. S^* is given in Figure 1 with multiplication given by Theorem 1(ii).

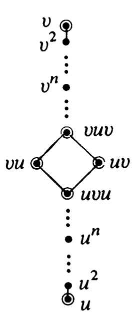

FIGURE 1

Theorem 1 and Corollary 3 hold, in particular, when either u, v or both u and v are idempotents. Thus if $u = u^2 \leqslant v = v^2$ then S^*, the po-subsemigroup generated by u and v, is an idempotent semigroup consisting of the six encircled elements in Figure 1. Clearly when S is linearly ordered then S^* consists of at most four elements (this is the case in [4] where S is a linearly ordered idempotent semigroup). We conclude this section with

COROLLARY 4.2. *Let S be a po-semigroup. Then $S \subseteq \hat{S}$ is a convex bounded po-subsemigroup of S iff $\hat{S} = [u,v]$ for some $u, v \in S$ such that $u \leqslant u^2 \leqslant v^2 \leqslant v$.*

3. In this section we apply the results of §2 to the bounded po-semigroup S. Since $0 \leqslant 0^2 \leqslant 1^2 \leqslant 1$, it follows by Theorem 1.2 that the elements 01, 10, $010 = 01 \wedge_E 10$ and $101 = 10 \vee_E 01$ are all idempotent, and for each $x \in S$

$$0x1 = 01, \qquad 1x0 = 10. \tag{3}$$

The following lemma is well known for po-groups.

LEMMA 1.3. *A cancellative po-semigroup cannot be bounded unless it is trivial.*

PROOF. Assuming S is both bounded and cancellative we get by (3) that $0^3 1 = 0^2 1$, $01^3 = 01^2$ implying that $0 = 01 = 1$.

The question of the existence of a zero element in a bounded po-semigroup is settled by

THEOREM 1.3. *Let S be a bounded po-semigroup. Then S has a zero element z iff* $01 = 10$, *and in that case* $z = 01$.

PROOF. If $z \in S$, then $01 \leqslant z1 = z = 0z \leqslant 01$ and so $z = 01$, while $10 \leqslant 1z = z = z0 \leqslant 10$ implies $z = 10$. Conversely, if $01 = 10$ holds, then for each $x \in S$, $01 = 0^2 1 \leqslant x01 \leqslant 101 = 01^2 = 01$, hence $x01 = 01$. $01x = 01$ follows similarly and $01 = z$ is the unique, two-sided zero element of S.

COROLLARY 1.3. *A commutative bounded po-semigroup contains a zero element.*

The following can easily be proved using (3).

LEMMA 2.3. *Let e denote any of the idempotents* 01, 10, 101 *or* 010 *in a bounded po-semigroup S. Then*: (i) *e is a primitive idempotent satisfying $eSe = e$*; (ii) *if $a = xey$ for some $x, y \in S$ then $a = a^2 = aSa$.*

$J \subseteq S$ is a (semigroup) *ideal* of S if $JS \subseteq J$, $SJ \subseteq J$. Let J_0, the *kernel* of S, denote the intersection of all ideals of S. S is (*zero*) *simple* [2] if S contains no proper ideal (except $\{z\}$ when $z \in S$).

THEOREM 2.3. *Let S be a bounded semigroup. Then J_0, the kernel of S, is nonempty and consists of all elements $x \in S$ satisfying $x = x^2 = xSx$. $J_0 = S01S = SxS$ for each $x \in J_0$.*

PROOF. If J is a nonempty ideal of S then $01 = 0J1 \subseteq SJS \subseteq J$ (see (3)). Thus $S01S \subseteq J$ for each ideal J and $J_0 = S01S$. By Lemma 2.3 we get that each $x \in J_0$ satisfies $x = x^2 = xSx$, and obviously $J_0 \subseteq SxS \subseteq S01S = J_0$. Conversely if $x = xSx$ then $x = x01x \in S01S = J_0$ which completes the proof.

Notice that by Theorem 1.3, $J_0 = \{z\}$ iff $01 = 10$.

COROLLARY 2.3. *Let $S = X^X$ where X is a bounded poset. Then $J_0 \approx X$ and consists of all constant maps on X.*

PROOF. In S, $0f = 0$ holds for each $f \in S$. Thus $J_0 = S01S = S0$ and $g \in J_0$ iff $g = f0$, i.e., $g(x) = f(0)$ for each $x \in X$. Hence J_0 consists of constant maps. Moreover, since $g = gSg$ holds for each constant map $g \in S$ the theorem follows using Theorem 2.3.

In case S is a complete po-semigroup we get:

THEOREM 3.3. *If S is a complete po-semigroup then J_0, the kernel of S, is a complete lattice in the order induced by S.*

PROOF. Let $\{x_j\} \subseteq J_0$ be given. If $y = \bigvee x_j$ then $y_0 = y01y \geqslant x_j01x_j = x_j$ for each j, so $y_0 \in J_0$ and $y_0 \geqslant \bigvee x_j = y$. If for some $x \in J_0$, $x \geqslant \bigvee x_j = y$, then $x = x01x \geqslant y01y = y_0$ follows and so $y_0 = \bigvee_{J_0} x_j$. The existence of $\bigwedge_{J_0} x_j$ is proved similarly.

Turning now to bounded po-semigroups which are (zero) simple we get using Theorem 2.3:

THEOREM 4.3. *Let S be a bounded po-semigroup without zero. Then S is simple iff S is an idempotent semigroup with $xSx = x$ for each $x \in S$, that is, iff S is a rectangular band.*

Let us now assume that S has a zero element ($z = 01 = 10$) and is zero-simple. Assuming $z \neq 1$ it follows (see [2, Lemma 2.28]) that $S = S1S$ hence $0 = x1y$ for some $x, y \in S$ and so $0 = x1y \geqslant 010 = z \geqslant 0$ implying $z = 0$. Similarly $z \neq 0$ would imply $z = 1$. In both cases we have

THEOREM 5.3. *Let S be a zero-simple bounded po-semigroup. Then for each $x \in S$ either $x^2 = x$ or $x^2 = z$ and $xSx = \{x, z\}$.*

PROOF. We can assume $z = 0 = 10 = 01$ and $z \neq 1$. Since $S = SxS$ for $x \neq z$,

$$(*) \qquad 1x1 = 1$$

is obvious. By $S = S1S$ we get that each ($z \neq$) $x \in S$ satisfies $x = a1b$ for some a, $b \in S$. Consequently $x^2 = a1ba1b$ equals either x (if $ba \neq z$) or z (if $ba = z$). $xSx = a1bSa1b = \{x, z\}$ follows by observing that $bSa \neq \{z\}$ when $b, a \neq z$ [2, Chapter 2] and by (*).

COROLLARY 3.3. *A (zero)-simple bounded po-semigroup S is completely (zero) simple.*

4. Here we show that analogues of Tarski's fixed point theorem hold in any complete po-semigroup S. If $x_0 s = s x_0 = x_0$ holds for some s, $x_0 \in S$ we say that x_0 *is fixed by* s. We now state

THEOREM 1.4. *Let S be a complete po-semigroup and let $s \in S$ be given. Then there exists an idempotent $x_0 \in S$ which is fixed by s. Moreover, E_s, the set of idempotents fixed by s is a complete lattice when ordered as a subset of S.*

This theorem will follow by

LEMMA 1.4. *Let S be a complete po-semigroup and let $s \in S$ be given. If $m \in S$ is increasing and $sm \geqslant m$, $ms \geqslant m$ then there exists $y_0 \in S$ such that* (i) $m \leqslant y_0$, (ii) $y_0 = y_0^2$, (iii) *y_0 is fixed by s and* (iv) *y_0 is the least element satisfying* (i), (ii), *and* (iii).

PROOF. Let X_s denote the set of decreasing elements $x_j \in S$ satisfying $m \leqslant x_j$, $sx_j \leqslant x_j$, $x_j s \leqslant x_j$. $X_s \neq \varnothing$ since $1 \in X_s$. Letting $y_0 = \wedge x_j$ we have

$$y_0^2 = (\wedge x_j)(\wedge x_j) \leqslant \wedge x_j^2 \leqslant \wedge x_j = y_0.$$

Obviously

$$m \leqslant \wedge x_j = y_0, \quad sy_0 = s(\wedge x_j) \leqslant \wedge sx_j \leqslant \wedge x_j = y_0,$$

and $y_0 s \leqslant y_0$ follows similarly, hence $y_0 \in X_s$. By $(y_0^2)^2 \leqslant y_0^2$ and $m \leqslant m^2 \leqslant y_0^2$ together with $sy_0^2 \leqslant y_0^2$ and $y_0^2 s \leqslant y_0^2$ we have $y_0^2 \in X_s$. Thus $y_0 \leqslant y_0^2$, and $y_0 = y_0^2 \in E$ follows. Obviously $m \leqslant ms \leqslant y_0 s$, and $(y_0 s)(y_0 s) \leqslant y_0^2 s = y_0 s$. Thus $y_0 s$ is decreasing. By $s(y_0 s) = (sy_0)s \leqslant y_0 s$ together with $(y_0 s)s \leqslant y_0 s$ one gets that $y_0 s \in X_s$ and so $y_0 \leqslant y_0 s$. $y_0 s = y_0$ follows and $sy_0 = y_0$ can be shown similarly. Consequently y_0 was shown to satisfy (i), (ii), (iii), and is by definition the least element having these properties.

PROOF OF THEOREM 1.4. Since $0 \leqslant 0^2$ with $0 \leqslant s0$, $0 \leqslant 0s$, Lemma 1.4 implies the existence of a minimal idempotent $y_0 \geqslant 0$ which is fixed by s. Obviously, $y_0 = 0_{E_s}$. For any set $\{y_j\} \subseteq E_s$ put $m = \vee y_j$. The fact that m is increasing with $m \leqslant ms$, $m \leqslant sm$ is easily checked. Lemma 1.4 applied again yields the existence of an idempotent y_0^* fixed by s such that $y_0^* = \vee_{E_s} y_j$. Thus E_s is a complete lattice.

Tarski's theorem can be even better "approximated" (Corollary 2.3) and actually generalized by

THEOREM 2.4. *Under the conditions of Theorem* 1.4 *the set $J_s \subseteq E_s$ of elements $y_0 \in J_0$ fixed by s is a nonempty complete lattice in the order induced by S.*

PROOF. Obviously (Theorem 2.3) $J_0 \subseteq E$ and for each $x \in E_s$, $x_0 = x01x \in J_0$ and x_0 is fixed by s. Thus $J_s \neq \varnothing$. For any set $\{x_j\} \subseteq J_s$ let $\hat{x} = \wedge_{E_s} x_j$ (Theorem 1.4). Then as in Theorem 3.3 one can easily show that $\hat{x}01\hat{x} = \wedge_{J_s} x_j$. The existence of $\vee_{J_s} x_j$ is proved analogously.

By similar methods another of Tarski's theorems [7] can be generalized:

THEOREM 3.4 *Let S be a complete po-semigroup and let $s_1 s_2 = s_2 s_1$ for some $s_1, s_2 \in S$. Then $E_{s_1} \cap E_{s_2}$ is a nonempty complete lattice.*

As an application of Theorem 2.4, we state

COROLLARY 1.4. *Let R be a ring and let $I_0 \subseteq R$ be a given (two-sided) ideal. Then the set of all ideals $I \subseteq R$ satisfying $I_0 I = I I_0 = I = I^2$ is a nonempty complete lattice when ordered by inclusion.*

REFERENCES

1. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R.I., 1967. MR **37** .

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1961. MR **24** #A2627.

3. L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, New York; Addison-Wesley, Reading, Mass., 1963. MR **30** #2090.

4. T. Merlier, *Nouvelles propriétés algébriques des demi-groupes idempotents totalement ordonnés*, C.R. Acad. Sci. Paris Sér. A.-B **277** (1973), A451–A452. MR **48** #3832.

5. O. Ore, *Galois connexions*, Trans. Amer. Math. Soc. **55** (1944), 493–513. MR **6**, 36.

6. Z. Shmuely, *Increasing and decreasing operators on complete lattices*, J. Combinatorial Theory Ser. A (to appear).

7. A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*. Pacific J. Math. **5** (1955), 285–309. MR **17**, 574.

DEPARTMENT OF MATHEMATICS, TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL