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FIBERED KNOTS IN HOMOTOPY 3-SPHERES

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ABSTRACT. Using the recently obtained result that each closed, orientable 3-manifold has a fibered knot, we exhibit a new equivalent of the 3-dimensional Poincaré conjecture.

F. González-Acuña has shown recently [5] that each closed, orientable 3-manifold M contains a *fibered knot*, that is, a tame knot K such that the exterior of K, $E(K)$, admits a fibration $E(K) \to S^1$ over a circle. Using this result, we can establish

THEOREM 1. *Let M be a homotopy 3-sphere. Suppose M has the property that for each fibered knot K in M there exists a tame knot L in the 3-sphere S^3 such that $\pi_1(M - K)$ is isomorphic to $\pi_1(S^3 - L)$. Then M is homeomorphic to S^3.*

This theorem can be viewed as a sharpening of a result of A. Connor and a converse of a theorem of L. Neuwirth. Connor's result [3] is similar to Theorem 1, but one needs to postulate that all the knot groups of M are "real" knot groups, not just those of fibered knots. Neuwirth shows (Theorem 9.2.3 of [7]) how, given a group G satisfying certain algebraic conditions, one can construct a homotopy 3-sphere with a fibered knot whose group is G; the theorem is stated in the form "... then, if the Poincaré conjecture is true, there exists a tame knot k in S^3 such that $\pi_1(S^3 - k) \approx G$". Our converse is that if each such group G is isomorphic to the group of a knot in S^3, then the Poincaré conjecture is true. Combining this with Neuwirth's theorem, we have

THEOREM 2. *The 3-dimensional Poincaré conjecture is equivalent to the conjecture that each group G with the properties listed below is isomorphic to the group of a tame knot in S^3.*

(1) *The commutator quotient $G/[G, G]$ is infinite cyclic.*

(2) *$[G, G]$ is a free group of rank $2n$.*

(3) *G has an element t whose normal closure ("consequence") is all of G.*

(4) *There is a free basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ for $[G, G]$ such that t commutes with the product of commutators $\prod_{i=1}^{n} [a_i, b_i]$.*

In the case where $n = 1$, Burde and Zieschang [2] establish the above

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conjecture by showing that G must be isomorphic to the group of a trefoil or figure-eight knot.

Our plan for proving Theorem 1 is to construct in the homotopy 3-sphere M a fibered knot $\hat{K}$ with enough special properties that if L is a knot in S^3 with $\pi_1(S^3 - L) \approx \pi_1(M - \hat{K})$, then such an isomorphism preserves enough of the geometry of $M - \hat{K}$ that we can conclude that M is homeomorphic to S^3.

Preliminaries. All spaces, subspaces, and maps considered here are polyhedral. If K is a knot in a homology 3-sphere M, and U is a regular neighborhood of K, the *exterior of* K, $E(K)$, is $M - \operatorname{int}(U)$. There is a pair of orthogonal simple closed curves μ, λ in ∂U such that μ bounds a disk in U, and therefore generates $H_1(E(K))$, and λ is null-homologous in $E(K)$. The curves μ and λ are unique up to orientations and ambient isotopy of ∂U. We call μ a *meridian* of K and λ a *longitude.* A simple closed curve in ∂U homologous to $p\mu + q\lambda$ is a (p, q)*-cable about* K. If K_1, K_2 are knots in M such that, for some 2-sphere S, $K_1 \cap S$ is an arc α, $K_2 \cap S = K_1 \cap S$, and S separates $K_1 - \alpha$ from $K_2 - \alpha$, then the knot $K_1 \cup K_2 - \operatorname{int}(\alpha)$, denoted $K_1 \# K_2$, is the *composition* of K_1 with K_2.

LEMMA. *Let M be a homology 3-sphere. If K is the composition $K_1 \# K_2$ of fibered knots in M then K is a fibered knot. If K is a (p,q)-cable $(q \neq 0)$ about a fibered knot K_0 then K is a fibered knot.*

PROOF. If $K = K_1 \# K_2$, we can choose regular neighborhoods such that $E(K)$ is the sum $E(K_1) \cup E(K_2)$, where $E(K_1) \cap E(K_2) = \partial E(K_1) \cap \partial E(K_2) = A$, an annulus whose center curve is a meridian of each of K_1, K_2. A fibration of $E(K)$ over S^1 can be constructed by adjusting the fibrations f_i: $E(K_i) \to S^1$ $(i = 1,2)$ so that $f_1|A = f_2|A$.

If K_0 is fibered and K is a (p,q)-cable about K_0, we can again explicitly construct a fibration of $E(K)$ (or use Stallings' characterization [10] of 3-manifolds that fiber over S^1). To see the fibering of $E(K)$, first push K into the interior of the regular neighborhood $U(K_0)$ and choose a regular neighborhood $U(K)$ contained in int $U(K_0)$. By considering how the exterior of a (p,q)-torus knot in S^3 fibers over S^1, we see that the manifold $W = U(K_0) - \operatorname{int} U(K)$ fibers over S^1 with fiber a connected 2-manifold F of genus $\frac{1}{2}(p-1)(q-1)$ having $(q+1)$ boundary components, one a longitude of K on $\partial U(K)$ and q components that are longitudes of K_0 on $\partial U(K_0)$. We next construct a new fibration for $E(K_0)$ by composing the given one with a q-fold covering map of $S^1 \to S^1$. The fibrations of $E(K_0)$ and W are then combined to give a fibration of $E(K)$ over S^1 with fiber a connected surface of genus $\frac{1}{2}(p-1)(q-1) + q$ (genus K).

PROOF OF THEOREM 1. Our proof is similar to [9] so we shall omit details and refer to [9] whenever practical.

Let M be a homotopy 3-sphere and suppose M has the property that the group of each fibered knot in M is isomorphic to the group of a knot in S^3.

By González-Acuña's theorem, there exists a fibered knot K in M. Let $\hat{K}$ be a $(1, 2)$-cable about the composition $K \# R$ of K with a trefoil knot R. We shall show that $E(K \# R)$ is homeomorphic to the exterior of a knot in S^3.

By the Lemma above, $\hat{K}$ is a fibered knot, so there exists a knot L in S^3 with $\pi_1(M - \hat{K}) \approx \pi_1(S^3 - L)$. Let $f\colon E(L) \to E(\hat{K})$ be a homotopy equivalence. The manifold $E(\hat{K})$ is the sum of a solid torus T and $E(K \# R)$, pasted together along an annulus A in their boundaries. For future reference, we note that each boundary component of A is a $(2, 1)$-cable about $K \# R$, so $\pi_1(A)$ does not contain an annihilator of $\pi_1(E(\hat{K}))$. As in [9, proof of Theorem 2], we can homotopically alter f so that $f^{-1}(A)$ is a finite collection $B_1, \ldots, B_n$ of essential annuli in $E(L)$. We show in the next three paragraphs that we may assume that $n = 1$ and $f|B_1$ is a homeomorphism.

Since A does not carry an annihilator of $\pi_1(E(\hat{K}))$, we have, as in [9, proof of Theorem 2, Claim 1], that each B_i separates $E(L)$ into a solid torus V_i and the exterior W_i of a nontrivial knot. Although we do not know enough about the knot L to proceed exactly as in [9], the proofs of Claims 2 and 3 can be modified to show that the annuli $B_1, \ldots, B_n$ can be ordered so that $W_1 \subset \cdots \subset W_n$ and $W_n \cap V_1$ is a solid torus with $(W_n \cap V_1, B_1, \ldots, B_n)$ homeomorphic to $(B_1 \times [1, n], B_1 \times \{1\}, \ldots, B_1 \times \{n\})$.

Let $g\colon E(\hat{K}) \to E(L)$ be a homotopy inverse of f. Then g can be homotopically adjusted so that $g^{-1}(B_1)$ is a collection $A_1, \ldots, A_m$ $(m \geqslant 1)$ of essential annuli in $E(\hat{K})$. By modifying Lemma 2.3 of [9] to handle properly embedded annuli in the exterior of a cable knot, we can show that, as loops, the components of ∂A_1 are homotopic in $\partial E(\hat{K})$ to the components of ∂A. If we consider the composition $f \circ g$, we then see that f can be adjusted so that for each i, $f|B_i$ is a homeomorphism of B_i onto A.

Now, as in Claim 6 of [9], we can use a "binding ties" argument to reduce the number of annuli B_i, eliminating them in pairs until $n \leqslant 1$. Since neither $\pi_1(T)$ nor $\pi_1(E(K \# R))$ generates $\pi_1(E(\hat{K}))$, we must have $n = 1$.

We complete the proof of Theorem 1 as in Claim 7 of [9]. The restriction $f_1 = f|W_1$ is a homotopy equivalence between W_1 and $E(K \# R)$ and maps B_1 homeomorphically onto A. Using the composite knot structure of $E(K \# R)$ we can (Lemma 2.3 of [9]) homotopically alter f_1 so that $f_1(\partial W_1) \subseteq \partial E(K \# R)$. By Theorem 6.1 of [11], W_1 is then homeomorphic to $E(K \# R)$. Since the proofs [1], [4], [6], [8] that composite knots have "Property P" do not depend on being in S^3, we conclude that a homeomorphism of $E(K \# R)$ to W_1 extends to a homeomorphism of M to S^3.

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