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TRIPLES OF 2×2 MATRICES WHICH GENERATE FREE GROUPS

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ABSTRACT. A theorem is proved which guarantees that certain triples of 2×2 matrices over the complex numbers freely generate a free group. (There is no a priori reason that these matrices lie in a two generator matrix group.)

1. **Introduction.** In the authors' work on automorphism groups the question often arises as to whether certain matrix groups are free or contain free subgroups, cf. [1]. For a concrete application to automorphism groups of the results derived here we refer the reader to §6 of [2]. In that paper, Chein proves that a certain triple of automorphisms of a free group of rank 3 are themselves free generators of a free group by computing a representation of that particular group into $\mathrm{SL}_2(Z)$ and then establishing that the three elements of $\mathrm{SL}_2(Z)$ are generators of a free group. The three elements are contained in the group M generated by $\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$, and so Chein was able to use the well-known theorem (Magnus [4], Sanov [5]) that M is a free group. Subsequently, there have been many generalizations of this theorem (see Lyndon and Ullman [3] and references therein), but all the generalizations concern pairs of 2×2 matrices over the complex numbers $\mathbf{C}$. This paper is an attempt to initiate a study of free subgroups of $\mathrm{SL}_2(\mathbf{C})$ of rank 3 which are not contained in (known) free subgroups of rank 2. In the applications to automorphism groups this, of course, will be an advantageous situation. Specifically, we prove

THEOREM. *Let x, y, z be complex numbers with norms greater than or equal to $M = 4.45$. Then, the matrices*

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 - z & -z \\ z & 1 + z \end{pmatrix}$$

are free generators of a free group.

If, in the matrices A, B, C of the theorem, we take $x = y = z = 2$, we will show that $G = \mathrm{gp}\langle A, B, C\rangle$ is a direct product of a free group of rank 2 and a group of order 2. In particular, G is not free, and so the norms of the

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numbers x, y, z must be larger than two. However, we show that if $x = y = z = 4$, then G is free of rank 3, although contained in a free subgroup of rank 2. Thus, the given value $M = 4.45$ may not be the best possible for G to be free of rank 3, although it might conceivably be sufficient (and perhaps best possible) for G to be free and not contained in a free subgroup of rank 2. We raise here the general question suggested by the theorem of whether there are any known nonabelian free subgroups of $SL_2(\mathbf{C})$ not contained in a free subgroup of rank 2.

If we conjugate the matrices in the theorem by $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $d \neq 0$, we have

COROLLARY. *Let x, y, z, u, v be complex numbers such that $|xy| \geqslant (4.45)^2$, $|z| \geqslant 4.45$, and $uv = -z^2$. Then,*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad \begin{pmatrix} 1+z & u \\ v & 1-z \end{pmatrix}$$

are free generators of a free group.

2. Proof of the theorem.

LEMMA 1. *Let $\alpha > 1$ and μ a complex number satisfying*

$$|\mu| \geqslant (\alpha^3 + 1)/(\alpha^2 - \alpha).$$

Let a, b, c be complex numbers such that either $|a| \geqslant \alpha|b| \geqslant \alpha^2|c|$ or $|a| \geqslant \alpha|c| \geqslant \alpha^2|b|$. Put $b' = b - \mu a + \mu c$. Then $|b'| \geqslant \alpha|a|$.

PROOF. Suppose first that $|a| \geqslant \alpha|b| \geqslant \alpha^2|c|$. Then

$$|b'| \geqslant (|\mu|(1 - 1/\alpha^2) - 1/\alpha)|a|,$$

and the conclusion follows provided $|\mu| \geqslant u = (\alpha^3 + \alpha)/(\alpha^2 - 1)$. Similarly, if $|a| \geqslant \alpha|c| \geqslant \alpha^2|b|$, the conclusion follows provided

$$|\mu| \geqslant v = (\alpha^3 + 1)/(\alpha^2 - \alpha).$$

Because $\alpha > 1$, we have $u < v$ and the conclusion holds if $|\mu| \geqslant v$. □

For the applications we want the lower bound on $|\mu|$ as small as possible. The minimum of v, for $\alpha > 1$, is attained at $\alpha = 2.33\cdots$, and, to two decimal places, is equal to $M = 4.45$.

LEMMA 2. *Let*

$$\alpha > 1, \quad M \geqslant \frac{\alpha^3 + 1}{\alpha^2 - \alpha}, \quad B = \begin{pmatrix} 1 & -y & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix}$$

where y is a complex number such that $|y| \geqslant M$. Let $V_A = \{(a,b,c) \mid |a| \geqslant \alpha|c| \geqslant \alpha^2|b| \text{ or } |a| \geqslant \alpha|c| \geqslant \alpha^2|b|\}$. In an analogous manner, define V_B and V_C as subsets of 3-dimensional vector space over $\mathbf{C}$. Then, for a nonzero integer k, $V_A B^k \subseteq V_B$ and $V_C B^k \subseteq V_B$.

PROOF. The first inclusion is a restatement of Lemma 1, while the second follows by symmetry. □

There are symmetrical forms of Lemma 2, where B^k is replaced by A^k or C^k which we use in the next lemma.

LEMMA 3. *Let*

$$\overline{A} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 1 & -y & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix}, \quad \overline{C} = \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ -z & 0 & 1 \end{pmatrix}$$

where x, y, z are complex numbers such that $|x|$, $|y|$, $|z|$ are all at least 4.45. *Then the group generated by $\overline{A}$, $\overline{B}$, $\overline{C}$ is a free group with $\overline{A}$, $\overline{B}$, $\overline{C}$ as free generators.*

PROOF. Let $W = X_1^{k_1} \cdots X_n^{k_n}$, $n \geqslant 1$, where each $X_i \in \{\overline{A}, \overline{B}, \overline{C}\}$, $X_i \neq X_{i+1}$, and each $k_i \neq 0$. Choose $X \neq X_1, X_n$. Then

$$V_X X_1^{k_1} \subsetneq V_{X_1}, \quad , \quad V_X X_1^{k_1} X_2^{k_2} \subsetneq V_{X_1} X_2^{k_2} \subsetneq V_{X_2},$$

and, continuing in this manner, $V_X W \subsetneq V_{X_n}$. Since $V_X \cap V_{X_n} = \varnothing$, $W \neq 1$. □

THEOREM. *If x, y, z are complex numbers each having norm at least* 4.45, *then the matrices*

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 - z & -z \\ z & 1 + z \end{pmatrix}$$

generate a free group of rank 3.

PROOF. Let $\overline{G}$ be the group generated by $\overline{A}$, $\overline{B}$, $\overline{C}$ in Lemma 3. Then the group

$$\hat{G} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overline{G} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

is isomorphic to $\overline{G}$ and hence is free. An element of $\hat{G}$ has the form

$$\hat{g} = \begin{pmatrix} 1 & 0 \quad 0 \\ * & \\ * & g \end{pmatrix}$$

where g is the lower 2×2 block of complex numbers and hence the mapping of $\hat{G}$ into $\mathrm{SL}_2(\mathbf{C})$ given by $\hat{g} \to g$ is a homomorphism. The kernel of this mapping consists of matrices of $\hat{G}$ of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$$

and, as a normal abelian subgroup of the free group $\hat{G}$, is trivial. A, B, C are the images of $\overline{A}$, $\overline{B}$, $\overline{C}$, repsectively, in this isomorphism and so the theorem follows. $\square$

3. **An example.** In this section we let

$$U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

and G be the group generated by U, V, W.

PROPOSITION. *G is isomorphic to the direct product of a free group of rank* 2 *and a group of order* 2.

PROOF. Since $V^{-1}U = -W$, G is generated by U, V, and $-I$. It is known that $F_2 = \text{gp}\,\langle U, V\rangle$ is free on U and V and hence does not contain the central involution $-I$. It follows that $G = F_2 \times C_2$ where $C_2 = \{I, -I\}$.

COROLLARY. *The group generated by*

$$U^2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad V^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad W^2 = \begin{pmatrix} 1-4 & -4 \\ 4 & 1+4 \end{pmatrix}$$

is a free group of rank 3.

PROOF. $\langle U^2, V^2, W^2\rangle = \langle U^2, V^2, (V^{-1}U)^2\rangle$ is a subgroup of the free group F_2, and clearly U^2, V^2, $(V^{-1}U)^2$ are a free set of generators. $\square$

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