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A CHARACTERIZATION FOR THE PRODUCTS OF k-AND-$\aleph_0$-SPACES AND RELATED RESULTS

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ABSTRACT. E. Michael introduced the notion of $\aleph_0$-spaces and characterized spaces which are both k-spaces and $\aleph_0$-spaces (or, briefly, k-and-$\aleph_0$-spaces) as being precisely the quotient images of separable metric spaces.

The purpose of this paper is to give a necessary and sufficient condition for the product of two k-and-$\aleph_0$-spaces to be a k-and-$\aleph_0$-space. Moreover, as related matters, we shall consider the products of k-spaces having other properties.

1. **Introduction**. Throughout this paper, we shall assume that all spaces are regular, and all maps are continuous surjection.

According to E. Michael [7], an $\aleph_0$-*space* is a space with a countable pseudobase. Here a collection $\mathcal{P}$ of subsets of X is a *pseudobase* for X if, whenever $C \subset U$ with C compact and U open in X, then $C \subset P \subset U$ for some $P \in \mathcal{P}$.

In [7], it was proved that any countable product of $\aleph_0$-spaces is an $\aleph_0$-space. However, as is well known, the product of two k-and-$\aleph_0$-spaces need not be a k-space [4, Example 1.11].

As for the product of k-and-$\aleph_0$-spaces, we have the following main theorem. First, we state a definition.

K. Morita [**11**] introduced the notion of the class $\mathfrak{S}'$. A space X is of *class* $\mathfrak{S}'$ if it is the union of countably many compact subsets X_n such that $A \subset X$ is closed whenever $A \cap X_n$ is closed in X_n for all n.

THEOREM 1.1. *Let X and Y be k-and-$\aleph_0$-spaces. Then $X \times Y$ is a k-and-$\aleph_0$-space if and only if one of the following three properties holds*:

(1) *X and Y are separable, metrizable spaces.*

(2) *X or Y is a locally compact, separable metrizable space.*

(3) *X and Y are spaces of class $\mathfrak{S}'$.*

As we shall see in §3, Theorem 1.1 can be extended to $\aleph$-spaces, a generalization of $\aleph_0$-spaces, which were introduced by P. O'Meara [17]. However, it cannot be extended to cosmic spaces in the sense of E. Michael [7]. As related matters, in §4, first, we shall give a characterization for the product of closed s-images of metric spaces to be a k-space. Second, we shall

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consider conditions for the product of spaces having other properties to be a k-space.

2. **Preliminary lemmas**. Following P. O'Meara [17], a collection $\mathcal{P}$ of subsets of a space X is a *k-network* for X if, whenever $C \subset U$ with C compact and U open in X, then $C \subset \bigcup \{F;\ F \in \mathcal{F}\} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{P}$.

An *$\aleph$-space*, according to O'Meara, is a space with a σ-locally finite k-network.

Clearly, metrizable spaces and $\aleph_0$-spaces are $\aleph$-spaces.

According to F. Siwiec [18], a space X is *strongly Fréchet* (= countably bisequential in the sense of E. Michael [9]) if, whenever $\{F_n;\ n = 1, 2, \ldots\}$ is a decreasing sequence accumulating at x in X, there exist $x_n \in F_n$ such that the sequence $\{x_n;\ n = 1, 2, \ldots\}$ converges to the point x.

Clearly, first-countable spaces are strongly Fréchet.

In [16], O'Meara proved that a first-countable $\aleph$-space is metrizable. We now show that this remains valid for a strongly Fréchet space.

LEMMA 2.1. *Let X be a strongly Fréchet $\aleph$-space. Then X is metrizable.*

PROOF. By the theorem in [16] quoted above, we need only prove X is first-countable.

Let $\mathcal{P}$ be a σ-locally finite (or merely point-countable) closed k-network for X. Suppose $x \in X$. Let $\mathcal{P}' = \{P \in \mathcal{P};\ x \in P\}$ and let $\mathcal{F}$ be the collection of all finite union of elements of $\mathcal{P}'$. Then the collection $\{\operatorname{int} F;\ F \in \mathcal{F}\}$ is a countable local base for x.

Indeed, for each open subset U containing x, let $\mathcal{P}'' = \{P \in \mathcal{P}';\ x \in P \subset U\}$. Then $\mathcal{P}''$ is a nonempty, countable subcollection of $\mathcal{P}'$. Let $\mathcal{P}'' = \{P_1, P_2, \ldots\}$. Let us show that $x \in \operatorname{int} \bigcup_{i=1}^n P_i \subset U$ for some n.

In fact, suppose that $x \not\in \operatorname{int} \bigcup_{i=1}^n P_i$ for each n. Let $F_n = X - \bigcup_{i=1}^n P_i$. Then $\{F_n;\ n = 1, 2, \ldots\}$ is a decreasing sequence accumulating at x. Since X is strongly Fréchet, there exist $x_n \in F_n$ such that the sequence $\{x_n;\ n = 1, 2, \ldots\}$ converges to the point x. The open subset U contains $\{x_n;\ n \geqslant m\} \cup \{x\}$ for some m. Let $K = \{x_n;\ n \geqslant m\} \cup \{x\}$. Then there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \bigcup \{F;\ F \in \mathcal{F}\} \subset U$. Some element F of $\mathcal{F}$ contains the point x. Then $F \in \mathcal{P}''$. Since F is a closed subset of X, there exists a subsequence of K in F. This is a contradiction to the choice of the sequence $\{x_n;\ n = 1, 2, \ldots\}$.

K. Morita [12] introduced the notion of M-spaces and characterized paracompact M-spaces as being precisely the perfect inverse images of metric spaces.

LEMMA 2.2. *Let X be an $\aleph$-space. If every closed subset of X which is a paracompact M-space is locally compact, then X has a σ-locally finite k-network consisting of compact subsets.*

PROOF. Let $\mathcal{P} = \bigcup_{n=1}^\infty \mathcal{P}_n$ be a σ-locally finite k-network for X. We assume that each element of $\mathcal{P}$ is closed, and for each n, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ and $\mathcal{P}_n$ is closed under finite intersections; that is, $\mathcal{P}_n$ contains all intersections of finitely many members of $\mathcal{P}_n$.

Let K be a compact subset of X. Let $\mathcal{P}' = \{P \in \mathcal{P}; P \cap K \neq \varnothing\}$, and let $\mathcal{K}$ be the collection of finite unions of elements of $\mathcal{P}'$ which contain the subset K. Then $\mathcal{K}$ is a nonempty, countable collection. Let $\mathcal{K} = \{P_1, P_2, \ldots\}$, and $K_n = \bigcap_{i=1}^{n} P_i$ for each n. Then, for each open subset U containing K, there is K_n with $K \subset K_n \subset U$. That is, the decreasing collection $\{K_n; n = 1, 2, \ldots\}$ of closed subsets is a countable local base for K.

Suppose that each K_n is not compact. Then each K_n is not countably compact. (Indeed, let some K_n be countably compact. Then K_n is first-countable, for each point of K_n is a G_δ-set in K_n. Thus, by the theorem of [16], K_n is metrizable. Then K_n is compact. This is a contradiction.) By the assumption, there is a closed subset $F = K \cup \bigcup_{n=1}^{\infty} D_n$ of X, where D_n is a countable, infinite discrete subset of K_n. Let Y be the quotient space obtained from F by identifying all points of K. Then Y is a countable, metrizable space which is not locally compact. Since F is the perfect inverse image of Y, it is a paracompact M-space which is not locally compact. This is a contradiction to the hypothesis in this lemma. Hence some K_m is compact, which implies all K_n $(n \geqslant m)$ are compact.

On the other hand, by the conditions on the collection $\mathcal{P}$, each K_n can be expressed as a union of finitely many elements of $\mathcal{P}$. Then, for each open subset U containing K, there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ such that each element of $\mathcal{F}$ is compact, and $K \subset \bigcup\{F; F \in \mathcal{F}\} \subset U$.

Hence, it follows that $\{P \in \mathcal{P}; P$ is compact$\}$ is a σ-locally finite k-network for X consisting of compact subsets, which completes the proof.

According to E. Michael [9, Theorem 7.3], a k-space in which every point is a G_δ-set is sequential in the sense of S. P. Franklin [4]. Then a k-and-$\aleph$-space is sequential, for each point of an $\aleph$-space is a G_δ-set. Hence, by the theorem in [9] and [20, Theorem 1.1], we have

LEMMA 2.3. *Let X be a k-and-$\aleph$-space, and let Y be first-countable. If $X \times Y$ is a k-space, then X is strongly Fréchet or Y is locally countably compact.*

Now, for the later convenience, we shall introduce the class $\mathfrak{T}'$ which is broader than the class $\mathfrak{S}'$. A space X is said to belong to the *class* $\mathfrak{T}'$ if it is the union of countably many closed and locally compact subsets X_n such that $A \subset X$ is closed whenever $A \cap X_n$ is closed in X_n for all n.

A Lindelöf space of the class $\mathfrak{T}'$ belongs to the class $\mathfrak{S}'$. A space of the class $\mathfrak{T}'$ is a k-space.

LEMMA 2.4. *Let X be a k-space. If X has a σ-locally finite k-network consisting of compact subsets, then X belongs to the class $\mathfrak{T}'$.*

PROOF. Let $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ ($\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all n) be a σ-locally finite k-network for X consisting of compact subsets. Let $P_n = \bigcup\{P; P \in \mathcal{P}_n\}$. Then each P_n is closed and locally compact. Since $\mathcal{P}$ is a k-network for X, each compact subset K of X is a subset of some P_n. Hence, it is easily seen that X belongs to the class $\mathfrak{T}'$.

The following lemma is proved as in the proof of [10, Lemma 2.1], so we shall omit the proof.

LEMMA 2.5. *Let X and Y belong to the class $\mathfrak{T}'$. Then $X \times Y$ also belongs to the class $\mathfrak{T}'$, and hence $X \times Y$ is a k-space.*

3. **Proof of the main theorem**. We shall prove the following generalization of the main theorem.

THEOREM 3.1. *Let X and Y be k-and-$\aleph$-spaces. Then $X \times Y$ is a k-and-$\aleph$-space if and only if one of the following three properties holds*:

(1) *X and Y are metrizable spaces.*

(2) *X or Y is a locally compact, metrizable space.*

(3) *X and Y are spaces of the class $\mathfrak{T}'$.*

PROOF. The "if" part is proved by [3, Theorem 3.2], Lemma 2.5, and by the fact that any countable product of $\aleph$-spaces is an $\aleph$-space (this is proved as in [7, Proposition 6.1]). So we shall prove the "only if" part.

Case 1. X and Y are strongly Fréchet: By Lemma 2.1, X and Y are metrizable.

Case 2. X is strongly Fréchet and Y is not strongly Fréchet: By Lemma 2.1, X is metrizable, hence it is locally compact by Lemma 2.3. Similarly, if X is not strongly Fréchet and Y is strongly Fréchet, then Y is locally compact, metrizable.

Case 3. Neither X nor Y is strongly Fréchet: Suppose that X contains a closed, paracompact M-space F which is not locally compact. Then F is the perfect inverse image of a metric space Z which is not locally compact. By assumption, $X \times Y$ is a k-space, then so the closed subset $F \times Y$ is also a k-space. Then $Z \times Y$ is a k-space, for it is the perfect image (hence, the quotient image) of the k-space $F \times Y$. Since Y is not strongly Fréchet, by Lemma 2.3, Z is locally compact. This is a contradiction. Hence X does not contain a closed, paracompact M-space which is not locally compact. Thus, by Lemma 2.2, X has a σ-locally finite k-network consisting of compact subsets. Hence, by Lemma 2.4, X belongs to the class $\mathfrak{T}'$. Similarly, Y belongs to the class $\mathfrak{T}'$. That completes the proof.

The following example shows that Theorems 1.1 and 3.1 become false if "$\aleph_0$-space" is weakened to "cosmic space".

Recall that a space is *cosmic* [7] if it has a countable network.

EXAMPLE 3.2. Let C be the compact subspace $[0, 1] \times \{0\}$ of the "butterfly space" S of L. F. McAuley; that is, of the plane with the usual topology at points not on the x-axis and with "bow-tie" neighborhoods of points on the x-axis. Let Y be the quotient space obtained from S by identifying all points of C. Then the cosmic space Y is neither locally compact nor first-countable [2, Remark 3.3]. Let $X = Y^\omega$, the product of countably many copies of Y. Then $X \times X$ is cosmic, and since it is the perfect image of the first-countable space S^ω, it is a k-space. However, X is not metrizable and moreover, as in the proof of [13, Theorem 1], it follows that X cannot be expressed as a countable union of closed, locally compact subsets. Thus the cosmic k-space X satisfies none of the three properties of Theorem 3.1.

4. **Some related results**. First, we shall consider products of closed s-images of metric spaces.

It appears to be unknown whether every closed s-image of a metric space is an $\aleph$-space. However, we have the following lemma.

Recall that a map $f\colon X \to Y$ is a *s-map* if each $f^{-1}(y)$ has a countable base.

LEMMA 4.1. *Let X be the image of a metric space Z under a closed s-map f. If every closed, metrizable subset of X is locally compact, then X is an $\aleph$-space.*

PROOF. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a σ-locally finite base for Z. We can assume that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, and that $\mathcal{B}_n$ is closed under finite intersections for each $n = 1, 2, \ldots$. Let $\mathcal{F}_n = \{f(\overline{B}); B \in \mathcal{B}_n\}$, and $\mathcal{C}_n = \{F \in \mathcal{F}_n; F$ is compact$\}$ for each $n = 1, 2, \ldots$. Then, as in the proof of Lemma 2.2, $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a σ-hereditarily closure-preserving closed network for X. Put $C_n = \bigcup \{C; C \in \mathcal{C}_n\}$.

We shall prove that each closed subset C_n of X is an $\aleph$-space. Let $x \in C_n$. Then, since f is a closed and s-map, there is an open subset V of C_n containing x which is covered by countably many elements $C_{n,i}$ of $\mathcal{C}_n$. Since the collection $\mathcal{V} = \{\overline{V} \cap C_{n,i};\ i = 1, 2, \ldots\}$ is a hereditarily closure-preserving closed covering of $\overline{V}$, it follows that each compact subset of $\overline{V}$ is covered by finitely many elements of $\mathcal{V}$. Also each element of $\mathcal{V}$ is compact, hence compact metric, and hence an $\aleph_0$-space. Thus it is easily checked that $\overline{V}$ is an $\aleph_0$-space, and hence so is V. This implies that C_n is locally an $\aleph_0$-space. On the other hand, C_n is a closed image of a metric space. Then, by [5], C_n is paracompact. Thus C_n is paracompact and locally an $\aleph_0$-space. Hence it follows that each C_n is an $\aleph$-space.

Now X is the closed image of a metric space, so by [6, Corollary 1.2] each compact subset of X is the image of some compact subset of Z. On the other hand, $\{B \in \mathcal{B};\ f(\overline{B}) \in \bigcup_{n=1}^{\infty} \mathcal{C}_n\}$ is an open covering of Z. Hence it follows that each compact subset of X is covered by finitely many closed $\aleph$-spaces C_n. This implies that X is an $\aleph$-space.

LEMMA 4.2 [9, COROLLARY 9.10]. *Let X be the closed image of a metric space. If X is strongly Fréchet, then it is metrizable.*

From the proof of Theorem 3.1, and from Lemmas 4.1 and 4.2, we have

THEOREM 4.3. *Let X and Y be the closed s-images of metric spaces. Then $X \times Y$ is a k-space if and only if one of the three properties of Theorem* 3.1 *holds.*

The question, however, remains whether this theorem is also valid with "s-image" weakened to "image".

Finally, we shall show that the proofs in §3 also give more information about the products of k-spaces having other properties. In fact, we have the following Theorems 4.4 and 4.5. The latter is a generalization of [20, Theorem 1.1].

Theorem 4.4 and the "only if" part of Theorem 4.5 are proved as in the proofs of Theorem 3.1 and Lemma 2.2, with Lemma 2.3 replaced by [20, Theorem 1.1]. As for Theorem 4.4, we make use of [15, Theorem 3.6], [14, Lemma 1.4], and the fact that every countably compact, strong Σ-space is compact (this is easily proved).

THEOREM 4.4. *Let X be a Fréchet space, or a k-space in which every point is a G_δ-set. Let Y be a σ-space (resp. a strong Σ-space in the sense of K. Nagami* [14]). *If $X \times Y$ is a k-space, then X is strongly Fréchet, or Y is the countable* *union of closed, locally compact, metric (resp. locally compact, paracompact) subsets.*

THEOREM 4.5. *Let X have the same properties as in Theorem* 4.4. *Let Y be a space of pointwise countable type in the sense of A. V. Arhangel'skiĭ* [1, *p.* 37]. *Then $X \times Y$ is a k-space if and only if X is strongly Fréchet, or Y is locally countably compact.*

The "if" part of this theorem is proved by [19, Corollary 2.4] and Proposition 4.6 below.

For the definitions of bi-k-spaces and countably bi-k-spaces, see [9, Definitions 3.E.1 and 4.E.1]. Obviously, spaces of pointwise countable type are bi-k, and strongly Fréchet spaces are countably bi-k.

PROPOSITION 4.6. *Let X be countably bi-k and let Y be bi-k. Then $X \times Y$ is a k-space.*

PROOF. Combining some known facts, this is proved step by step. By [9, Theorem 3.E.3], Y is a biquotient image of a paracompact M-space Z. Then, by [8, Theorem 1.2], $X \times Y$ is a biquotient image of $X \times Z$. We shall prove that $X \times Z$ is a k-space. By [12, Theorem 6.1], Z is a perfect inverse image of a metric space T. Thus $X \times Z$ is a perfect inverse image of $X \times T$. But $X \times T$ is a k-space by [9, Theorem 4.E.3]. Hence $X \times Z$ is a k-space by [1, Theorem 2.5]. Thus $X \times Y$ is a k-space, for it is the biquotient image (hence, the quotient image) of the k-space $X \times Z$.

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