

CHARACTERISTIC FUNCTIONS OF OPERATORS SIMILAR TO UNITARY OPERATORS

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ABSTRACT. Suppose T denotes an operator which is similar to a unitary operator and θ_T denotes its characteristic function—considered as an operator on $H^2(D_T)$. It is proved that θ_T is a bounded operator if and only if it is the inverse of a bounded operator.

In this note we study characteristic functions of operators which are similar to unitary operators. As seen earlier [3], invertibility of the characteristic function is a sufficient condition for a power-bounded operator to be similar to a unitary operator. Now we prove that if an operator is similar to unitary, then being bounded and being invertible are equivalent conditions for its characteristic function. Corollaries showing ways in which the conditions may fail are easily derived.

Following the notation in [1] we denote the characteristic function of T by θ_T . Specifically

$$\theta_T(\lambda) = [-TJ_T + \lambda Q_{T^*}(I - \lambda T^*)^{-1} Q_T] |\overline{Q_T \mathcal{K}}|$$

where $J_T = \text{sgn}(I - T^*T)$ and $Q_T =$ positive square root of $|I - T^*T|$. We denote by U the minimal J -unitary dilation of T and by U_+ the minimal J -isometric dilation of T . The residual space R and the shift-space M are defined in close analogy with the dilation theory for contractions. For explicit definitions and details see [1].

LEMMA 1. *Suppose T is an operator whose spectrum is contained in the unit circle. Then there exist bounded operators X, Y, Z, W acting on the appropriate defect spaces such that for $|\lambda| < 1$,*

- (i) $\theta_{T^{-1}}(\lambda) = X\theta_T^{-1}(\bar{\lambda})^* Y$ and
- (ii) $\theta_T^{-1}(\lambda) = Z\theta_{T^{-1}}(\bar{\lambda})^* W$.

For proof see [3, Lemma 2].

PROPOSITION 1. *Suppose $\sup_{|\lambda| < 1} \|\theta_T(\lambda)\| < \infty$ and T is similar to a unitary operator. Then $\sup_{|\lambda| < 1} \|\theta_T^{-1}(\lambda)\| < \infty$.*

PROOF. Following the notation in [1] we denote the residual space of U by

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R and the projection onto it by P_R . We first prove that $P_R|H$ is invertible. Suppose $\{h_n\} \subseteq H$, $\|h_n\| = 1$ and $\|P_R h_n\| \rightarrow 0$, i.e., $\|h_n - JP_{M_*} h_n\| \rightarrow 0$. If $\sup_{k \geq 0} \|T^{*-k}\| = 1/c$ and say $\sup_{k \geq 0} \|U_+^k\| = a$, then let $0 < \varepsilon < c/2a$. For the proof of power-boundedness of U_+ see [1, p. 137]. Let n_0 be such that $\|h_{n_0} - JP_{M_*} h_{n_0}\| < \varepsilon$, and let $k_{n_0} \in M_*$ be a finite sum of the form $k_{n_0} = \sum_{n=0}^{N_*} U_+^{n+1} h_{*n}^{(-1)}$ such that $\|P_{M_*} h_{n_0} - k_{n_0}\| < \varepsilon$. If $k > N_* + 1$ then

$$\begin{aligned} U_+^{*k} J k_{n_0} &= P_+ U_+^{*k} J \sum_{n=0}^{N_*} U_+^{n+1} h_{*n}^{(-1)} \\ &= P_+ J \sum_{n=0}^{N_*} U_+^{-k+n+1} h_{*n}^{(-1)} = 0 \end{aligned}$$

since $U^{-1} D_{T^*}^{(-1)} \perp K_+$. Thus choosing a fixed $k > N_* + 1$ we have

$$\begin{aligned} c &= c\|h_{n_0}\| \leq \|T^{*k} h_{n_0}\| = \|U_+^{*k} h_{n_0}\| = \|U_+^{*k} (h_{n_0} - J k_{n_0})\| \\ &\leq a\|h_{n_0} - J k_{n_0}\| \leq a\|h_{n_0} - JP_{M_*} h_{n_0}\| + a\|JP_{M_*} h_{n_0} - k_{n_0}\| \\ &< 2a\varepsilon < c, \end{aligned}$$

which is clearly a contradiction. Hence $P_R|H$ has a positive lower bound. That it has a dense range follows by repeating the argument given by Sz.-Nagy and Foiaş (see [4, p. 63]). $M = K_+ \ominus H$, $JH = H$, $R = K_+ \ominus JM_*$, it follows that $P_{M_*}|M$ is invertible. Since $F^* \theta \Phi = P_{M_*}|M$ it follows that θ is invertible.

THEOREM 1. *Suppose T is similar to a unitary operator. Then its characteristic function is bounded on the open unit disk if and only if it is boundedly invertible on it.*

PROOF. Suppose T is similar to unitary and $\sup_{|\lambda| < 1} \|\theta_T^{-1}(\lambda)\| < \infty$. Then by Lemma 1 the characteristic function of T^{-1} is bounded on the open disk. It follows by Proposition 1 that $\sup_{|\lambda| < 1} \|\theta_{T^{-1}}^{-1}(\lambda)\| < \infty$. Using Lemma 1 once again we see that $\sup_{|\lambda| < 1} \|\theta_T(\lambda)\| < \infty$. The converse is a restatement of Proposition 1.

REMARK. In the light of Sahnovič's theorem [1], Theorem 1 can be restated as

THEOREM 1'. *Any two of the following imply the third. (i) θ_T is bounded, (ii) θ_T is boundedly invertible, (iii) T is similar to a unitary operator.*

When combined with [3, Proposition 2] this theorem yields the following corollaries.

COROLLARY 1. *If an operator T is similar to a unitary operator, but not unitary, and if its spectrum has zero Lebesgue measure then $\sup_{|\lambda| < 1} \|\theta_T(\lambda)\| = \sup_{|\lambda| < 1} \|\theta_T^{-1}(\lambda)\| = \infty$.*

COROLLARY 2. *If an operator on a finite dimensional space is similar to a unitary operator but is not unitary then $\sup_{|\lambda| < 1} \|\theta_T(\lambda)\| = \sup_{|\lambda| < 1} \|\theta_T^{-1}(\lambda)\| = \infty$.*

COROLLARY 3 (SZ.-NAGY-FOIAŞ). *A contraction which is similar to a unitary*

operator and whose spectrum has zero measure is unitary.

Thus there are many examples showing that no one of the conditions in Theorem 1' alone implies the other two.

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