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A SUBALGEBRA CONDITION IN LIE-ADMISSIBLE ALGEBRAS

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The author wishes to present this paper in memory of his wife, Myung Mi Myung, whose untimely death occurred during the preparation of this paper. She was a trained mathematician and unselfishly encouraged the author during her illness and assisted in the preparation of the manuscript.

ABSTRACT. Let A be a finite-dimensional, flexible, Lie-admissible algebra over a field Φ of characteristic $\neq 2$. Let S be a subalgebra of A^- and H be a Cartan subalgebra of S. It is shown that S is a subalgebra of A if and only if $HH \subseteq S$.

For an algebra A, denote by A^- the algebra with multiplication $[x,y] = xy - yx$ defined on the vector space A. If A^- is a Lie algebra then A is said to be Lie-admissible. If A is, in addition, a finite-dimensional flexible algebra over a field Φ then a Cartan subalgebra of A^- has played an important role for the structure of the algebra A [1], [3]. Let S be a subalgebra of the Lie algebra A^-. In this note, we give a condition in terms of a Cartan subalgebra of S that S be a subalgebra of A.

THEOREM. *Let A be a finite-dimensional, flexible, Lie-admissible algebra over a field Φ of characteristic $\neq 2$. Let S be a subalgebra of the Lie algebra A^- and H be a Cartan subalgebra of S. Then S is a subalgebra of A if and only if $HH \subseteq S$.*

PROOF. One can assume that Φ is algebraically closed. Since A is flexible and Lie-admissible, the mapping $\operatorname{ad} x$: $a \to [a,x]$ is a derivation of A for all $x \in A$; that is, $[a,bc] = [a,b]c + b[a,c]$ for all a, $b \in A$. Let $x, y \in A$. Then the flexible law $(x,y,y) + (y,y,x) = 0$ implies $y(yx) = (xy)y - [x,y^2]$. Hence we get

$$[[x,y],y] = (xy)y - (yx)y - y(xy) + y(yx) = (xy)y - 2y(xy) + y(yx)$$
$$= 2(xy)y - [x,y^2] - 2y(xy) = 2[xy,y] - [x,y^2] = 2[x,y]y - [x,y^2].$$

Therefore, we have $[x,y]y = \frac{1}{2}([x,y^2] + [[x,y],y])$.

Let $S_\alpha = \{x \in S \mid x(\operatorname{ad} h - \alpha(h)I)^n = 0, h \in H,$ for some $n > 0\}$. Then since H is a Cartan subalgebra of S, we have the root space decomposition

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$S = \sum_\alpha S_\alpha$ of S relative to H, where $H_0 = S$ and $[S_\alpha, H] \subseteq S_\alpha$. For $\alpha \neq 0$, we first show that $S_\alpha H \subseteq S$. Since $\alpha \neq 0$, there exists an element $h \in H$ such that $\alpha(h) \neq 0$. Then $\operatorname{ad} h$: $S_\alpha \to S_\alpha$ is surjective. For, if $[x, h] = 0$ for some $x \neq 0$ in S_α then $x(\operatorname{ad} h - \alpha(h)I)^n = 0$ implies that $\alpha(h)^n x = 0$ and so $\alpha(h) = 0$. Now, for every element $x \in S_\alpha$, we have

$$[x, h]h = \tfrac{1}{2}([x, h^2] + [[x, h], h]) \in [S, HH] + [[S, H], H] \subseteq S.$$

Since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha h \subseteq S$. Let k be any element in H. Then we have

$$\begin{aligned}[x, h]k &= [x, hk] - h[x, k] = [x, hk] - [h, [x, k]] - [x, k]h \\ &\in [S, HH] + [H, [S, H]] + [S_\alpha, H]h \subseteq S + S_\alpha h \subseteq S.\end{aligned}$$

Again, since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha H \subseteq S$ for $\alpha \neq 0$. Also, $S_0 H = HH \subseteq S$ and so we have that $SH \subseteq S$.

For any $\alpha \neq 0$, let h be an element in H such that $\alpha(h) \neq 0$. Let $x \in S_\alpha$, $y \in S$. Then

$$y[x, h] = [x, yh] - [x, y]h \in [S, SH] + SH \subseteq S.$$

Since $[S_\alpha, h] = S_\alpha$, this shows that $SS_\alpha \subseteq S$ for $\alpha \neq 0$. From $SS_0 = SH \subseteq S$, we have that $SS \subseteq S$, as required.

If S is a subalgebra of A^- which is classical in the sense of Seligman [4], then, in view of [3, Corollary 3.4], the theorem enables us to give a condition that S is a Lie algebra under the multiplication in A, so that a classical Lie algebra is imbedded into A as a subalgebra. An element $x \in A$ is called nilpotent if x is power-associative and $x^n = 0$ for some $n > 0$. We also say that a subset M of A is nil if every element of M is nilpotent. The following is an immediate consequence of the theorem and [3, Corollary 3.4].

COROLLARY 1. *Let S be a subalgebra of A^- which is classical and H be a classical Cartan subalgebra of S. Then S is a Lie algebra under the multiplication in A if and only if $HH \subseteq S$ and H is nil in A.*

In particular, if A is power-associative and A^- is semisimple over Φ of characteristic 0, it is shown that A is a nilalgebra [2] and turns out to be a Lie algebra [1], [3]. The original proof of this requires that Φ is algebraically closed; however, if Φ is not algebraically closed, it can be extended to its algebraic closure. Therefore, we have

COROLLARY 2. *Let Φ be of characteristic 0 and let S be a semisimple subalgebra of A^-. Suppose that every element of S is power-associative. Then S is a Lie algebra under the multiplication in A if and only if S contains a Cartan subalgebra H such that $HH \subseteq S$.*

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REFERENCES

1. P. J. Laufer and M. L. Tomber, *Some Lie admissible algebras*, Canad. J. Math. **14** (1962), 287–292. MR **25** #104.

2. H. C. Myung, *A remark on the proof of a theorem of Laufer and Tomber*, Canad. J. Math. **23** (1971), 270. MR **42** #4602.

3. ———, *Some classes of flexible Lie-admissible algebras*, Trans. Amer. Math. Soc. **167** (1972), 79–88. MR **45** #3489.

4. G. B. Seligman, *Modular Lie algebras*, Ergebnisse Math. Grenzgebiete, Band 40, Springer-Verlag, New York, 1967. MR **39** #6933.

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