

TRUTH AND INFINITY¹

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ABSTRACT. This paper formalizes Gödel's 1946 conjecture that every set-theoretic sentence is decidable from the present axioms plus some true axioms of infinity; and we prove a weak variant of this conjecture to be true for every ZF universe. We then make precise the extent to which unbound quantifiers can be taken to range only over ordinals in ZF, obtaining a sort of normal-form theorem. The last section relates these results to the problem of how wide the class of all sets is.

1. This paper is based on Gödel's 1946 remarks about the possible termination of the process of finding stronger set theories. Very little has been written about these remarks (see, however, Takeuti [4]); hopefully this paper will provoke further discussion.

In §2 we reprint most of the relevant passage from Gödel [1] and attempt to formalize the two conjectures there presented.

§3 is devoted to a proof of a version of the stronger of these conjectures: that every ZF-sentence is decidable from ZF plus some true axioms of infinity. We hasten to point out that our proof is possible only because of a very generous interpretation of the notion "axiom of infinity".

In §4 we use this result to obtain a normal form theorem for ZF-sentences.

In §5 we assess our results and discuss the import of Gödel's conjecture. The moral that the class of all sets should not be a model of the power-set axiom is drawn.

One of the results of §3 was announced in [2]. The proof of the Lemma in §3 has been corrected at the referee's suggestion.

2. All formulae and sentences are to be in the language of ZF. We will avoid some notational complications by not always maintaining the distinction between a sentence and its Gödel number. If x is a set or class, $Th(x)$ is the set of all sentences true in the structure $\langle x, \varepsilon_x \rangle$. $x \models \phi[a_1, \dots, a_n]$ means that $a_1, \dots, a_n \in x$ and that the sequence $\langle a_1, \dots, a_n \rangle$ (abbreviated as \vec{a}) satisfies ϕ in the structure $\langle x, \varepsilon_x \rangle$. $D(x)$ is the set of all pairs $\langle \vec{\phi}, \vec{a} \rangle$ such that $x \models \phi[\vec{a}]$.

Let us begin by quoting from Gödel [1], where we have italicized a definition and two conjectures:

"It is certainly impossible to give a combinatorial and decidable characteriza-

Presented to the Society, February 15, 1973; received by the editors September 6, 1974 and, in revised form, April 21, 1975.

AMS (MOS) subject classifications (1970). Primary 02K15, 04A10; Secondary 02A05.

Key words and phrases. Gödel's conjecture, axioms of infinity, class of all sets, reflection principle, power-set axiom.

¹ This paper was written with the support of a SUNY/Research Foundation grant.

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tion of what an axiom of infinity is but there might exist, e.g. a characterization of the following sort: *An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true.* Such a concept of demonstrability might have the required closure property, i.e. the following could be true: *Any proof for a set-theoretic theorem in the next higher system above set theory (i.e. any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity.* It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that *every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets.* ”

Let M be a set large enough to be sufficiently like the class of all sets. Gödel suggests that there is a decidable (say r.e.) set PI of possible axioms of infinity, and that we can take the set of axioms of infinity to be $I = PI \cap Th(M)$.

A possible formulation of Gödel's first conjecture might be:

(G1) For any sentence σ , $MK \cup \{\psi^{(V)} : \psi \in I\} \vdash \sigma^{(V)}$ iff $ZF \cup I \vdash \sigma$.

The idea behind (G1) is the following. If we have a ZF-definable set $\Sigma \subseteq PI$, and a θ such that $R_\theta \models \Sigma$, then the fact that such a θ exists either (a) adds nothing to our first-order knowledge of V , or (b) can be put in the form “there exists a θ satisfying such and such an axiom of infinity”.

It would seem that (G1) will hold provided that the existence of a natural model of each ZF-definable subsystem of $ZF \cup I$ is implied by some sentence in I .

We can get different versions of (G1) according to what kind of a theory we put over our “class of all sets” M . That is, in (G1) MK could be replaced, for instance, by Ackermann's set theory, Takeuti's Nodal Transfinite Type theory, or Reinhardt's system S^+ .

It is easier to formalize and discuss Gödel's second conjecture:

(G2) $ZF \cup I$ is a complete theory.

3. Clearly (G2) implies (G1).

Under what assumptions on M and PI will $ZF \cup (PI \cap Th(M))$ be complete?

All we will require of M is that it be a model of ZF. How shall we choose PI ? There are two cautions.

First, PI should contain arbitrarily long sentences. For if there is an upper bound to the length of the sentences in PI , then PI and I are finite. If I is finite, then it is equivalent to a single ZF-sentence, and $Con(ZF \cup I)$ will be a ZF-sentence as well. Now $ZF \cup I$ is consistent since it holds in M , so by the incompleteness theorem $ZF \cup I$ does not decide $Con(ZF \cup I)$ —implying that (G2) fails. Note that this argument will not work in the general case where there is no ZF-term representing I , for then $Con(ZF \cup I)$ is not a ZF-sentence.

Second, PI should not consist only of sentences whose unbound quantifiers range over the ordinals. For such sentences relativize to L , and we believe that “There is a measurable cardinal” is an axiom of infinity.

DEFINITION. σ is an O -sentence iff for some formula ϕ , σ is a sentence of the form $(Q_1 \beta_1) \cdots (Q_n \beta_n)(\exists R_\alpha)[R_\alpha \models \phi[\beta_1, \dots, \beta_n]]$, where each Q_i is an \forall or an \exists .

All the sentences normally called axioms of infinity are O -sentences. E.g. "there are arbitrarily large inaccessible cardinals" $\leftrightarrow (\forall \beta)(\exists R_\alpha)[R_\alpha \models \text{"there is an inaccessible cardinal larger than } \beta"]$, "there is a measurable cardinal" $\leftrightarrow (\exists R_\alpha)[R_\alpha \models \text{"there is a measurable cardinal"}]$, "there is a compact cardinal" $\leftrightarrow (\exists \beta_1)(\forall \beta_2)(\exists R_\alpha)[R_\alpha \models \text{"}\beta_1 \text{ is a compact cardinal less than } \beta_2\text{"}]$.

Let OS be the set of O -sentences.

THEOREM 1. *If M is a model of ZF , then $ZF \cup (OS \cap Th(M))$ is complete.*

Theorem 1 is a direct consequence of the lemma proved below.

DEFINITION. ψ is an R -formula iff for some formula ϕ with no unbound quantifiers, ψ has the form $(Q_1 R_{\beta_1}) \cdots (Q_n R_{\beta_n})\phi[R_{\beta_1}, \dots, R_{\beta_n}]$. An R -sentence is an R -formula which is a sentence.

LEMMA. (i) *Every R -sentence is equivalent in ZF to some O -sentence.*

(ii) *Every ZF -sentence is equivalent in ZF to some R -sentence.*

PROOF.

$$\begin{aligned} \text{(i)} \quad & (Q_1 R_{\beta_1}) \cdots (Q_n R_{\beta_n}) \psi[R_{\beta_1}, \dots, R_{\beta_n}] \\ & \leftrightarrow (Q_1 \beta_1) \cdots (Q_n \beta_n)(\exists R_\alpha)[R_\alpha \models (\exists v_1, \dots, v_n)[v_1 = R_{\beta_1} \wedge \cdots \\ & \qquad \qquad \qquad \wedge v_n = R_{\beta_n} \wedge \psi]], \end{aligned}$$

since ψ has no unbound quantifiers.

(ii) If σ is a ZF -sentence, then there is a quantifier-free formula ψ such that $\sigma \leftrightarrow (Q_1 v_1) \cdots (Q_n v_n)\psi$. We will be done if we can prove

$$\begin{aligned} (*) \quad & (Q_1 v_1) \cdots (Q_n v_n)\psi \\ & \leftrightarrow (Q_1 R_{\beta_1}) \cdots (Q_n R_{\beta_n})(Q_1 v_1 \in R_{\beta_1}) \cdots (Q_n v_n \in R_{\beta_n})\psi. \end{aligned}$$

(*) will follow if we can prove in ZF that for any quantifier free ϕ ,

$$\begin{aligned} & (\exists x)(Q_1 R_{\beta_1}) \cdots (Q_n R_{\beta_n})(Q_1 v_1 \in R_{\beta_1}) \cdots (Q_n v_n \in R_{\beta_n})\phi \\ (**) \quad & (\exists R_\alpha)(Q_1 R_{\beta_1}) \cdots (Q_n R_{\beta_n})(\exists x \in R_\alpha)(Q_1 v_1 \in R_{\beta_1}) \cdots (Q_n v_n \in R_{\beta_n})\phi. \end{aligned}$$

(*) follows inductively from (**) since replacing ϕ by $\sim\phi$ and replacing each side of (**) by its negation yields (**) with " \exists " replaced by " \forall ".

Now we prove (**). The idea is that we replace " $(\exists x)$ " by " $(\exists R_\alpha)$ " " $(\exists x \in R_\alpha)$ " and then move the $(\exists x \in R_\alpha)$ in past the $(Q_i R_{\beta_i})$ one at a time. One can always interchange two existential quantifiers, so the only problem is to show, for any k between 1 and n such that Q_k is \forall , that if we fix $R_\alpha, R_{\beta_1}, \dots, R_{\beta_{k-1}}$ then

$$\begin{aligned}
 & (\exists x \in R_\alpha)(\forall R_{\beta_k}) \cdots (Q_n R_{\beta_n}) \\
 (***) \quad & (Q_1 v_1 \in R_{\beta_1}) \cdots (\forall v_k \in R_{\beta_k}) \cdots (Q_n v_n \in R_{\beta_n}) \phi \\
 & \leftrightarrow (\forall R_{\beta_k})(\exists x \in R_\alpha) \cdots (Q_n R_{\beta_n}) \\
 & (Q_1 v_1 \in R_{\beta_1}) \cdots (\forall v_k \in R_{\beta_k}) \cdots (Q_n v_n \in R_{\beta_n}) \phi.
 \end{aligned}$$

The forward implication is obvious. To prove the reverse implication, assume the right-hand side of (***). Now for each value of β_k , define the set

$$\begin{aligned}
 s(\beta_k) = \{x \in R_\alpha : (Q_{k+1} R_{\beta_{k+1}}) \cdots (Q_n R_{\beta_n})(Q_1 v_1 \in R_{\beta_1}) \\
 \cdots (\forall v_k \in R_{\beta_k}) \cdots (Q_n v_n \in R_{\beta_n}) \phi\}.
 \end{aligned}$$

Since we have assumed the right-hand side of (***), each $s(\beta_k)$ is nonempty. Note also that $\beta_k < \beta'_k \rightarrow s(\beta_k) \supseteq s(\beta'_k)$. This is true since the only appearance of β_k in the specification condition of $s(\beta_k)$ is in the part “ $(\forall v_k \in R_{\beta_k})$ ”, so the smaller β_k is, the easier it is for an x to satisfy the specification condition of $s(\beta_k)$.

Now, if there is an $x \in \bigcap_{\beta_k} s(\beta_k)$, then this x will witness the truth of the left-hand side of (***). We can see, however, that this intersection is nonempty, for if it were empty, the map from R_α into On defined by $f'x = (\mu\beta_k)[x \notin s(\beta_k)]$ would be cofinal in On , contradicting the axiom of replacement. In other words, since $\{s(\beta_k) : \beta_k \in On\}$ is a \supseteq -sequence of nonempty sets it must be eventually constant, and must thus have nonempty intersection.

So now we have proved (***) ; so we have (**), (*), and part (ii) of our Lemma. \square

4. The Lemma has a certain intrinsic interest as a sort of normal form theorem for ZF. One can sharpen it.

THEOREM 2. *There is a Δ_2^{ZF} formula $\tau(v_0, v_1)$ such that for every ZF-sentence σ there are integers s and n and a sequence Q_1, \dots, Q_{n+1} of quantifiers such that*

$$ZF \vdash \sigma \leftrightarrow (Q_1 \beta_1) \cdots (Q_{n+1} \beta_{n+1}) \tau[s, \langle \beta_1, \dots, \beta_{n+1} \rangle].$$

PROOF. By the Lemma we can assume that σ is an O -sentence which we abbreviate as $(\vec{Q}\beta)(\exists\alpha)[R_\alpha \models \psi[\vec{\beta}]]$.

Note that

$$\sigma \leftrightarrow \begin{cases} (\vec{Q}\beta)(\exists\alpha)(\forall r)[r \neq R_\alpha \vee \langle \ulcorner \psi \urcorner, \vec{\beta} \rangle \in D(r)] \\ (\vec{Q}\beta)(\exists\alpha)(\exists r)[r = R_\alpha \wedge \langle \ulcorner \psi \urcorner, \vec{\beta} \rangle \in D(r)] \end{cases}$$

It is not hard to see that $r = R_\alpha$ is a Δ_2^{ZF} predicate of r and α . As is well-known, $\langle \ulcorner \psi \urcorner, \vec{\beta} \rangle \in D(r)$ is a Δ_1^{ZF} predicate of $\ulcorner \psi \urcorner$, $\vec{\beta}$ and r . Thus we can see that there is a Δ_2^{ZF} predicate τ' of $\ulcorner \psi \urcorner$, $\vec{\beta}$, and α such that $\sigma \leftrightarrow (\vec{Q}\beta)(\exists\alpha)\tau'[\ulcorner \psi \urcorner, \vec{\beta}, \alpha]$.

Viewing “ $(\exists\alpha)$ ” as “ $(Q_{n+1} \beta_{n+1})$ ”, we obtain our result. \square

Note that in Theorem 2, s , n , and the sequence Q_1, \dots, Q_{n+1} are recursive in “ σ ”.

5. It could reasonably be argued that Theorem 1 is not an acceptable answer to (G2) since the set of O -sentences does not seem to organically embody the principles by which possible axioms of infinity are obtained, e.g. reflection principles, the principle of analogy to ω , considerations of imaginary extensions of the class of all sets, etc. In particular, it does not seem fair to call the O -sentence $(\exists R_\alpha)[R_\alpha \models 2^{\aleph_0} = \aleph_1]$ an axiom of infinity.

On the other hand, if we view the class of all sets as the limit of the R_α sequence, the basic form of the O -sentences seems correct for axioms of infinity—a string of unbounded ordinal quantifiers followed by the assertion of the existence of a partial universe R_α satisfying a certain formula with ordinal parameters. The O -sentence formulation strikes us as incorrect only after we realize that *every* ZF-sentence is equivalent to an O -sentence.

It may be that for any *constructive* set PI of possible axioms of infinity, either (i) (G2) holds, but only because every ZF-sentence is provably equivalent to some sentence in PI , or (ii) (G2) fails because consideration of the true members of PI leads one to conceive of a true axiom of infinity which is not in PI . The problem could be that the notion “true axiom of infinity” is not reducible to the notion of truth in the class of all sets plus some constructive notion of possible axioms of infinity... the notion “possible axiom of infinity” may itself be inherently nonconstructive.

It is, however, the case that certain ZF-sentences, e.g. “there is a Mahlo cardinal”, become accepted as true axioms of infinity. One can ignore the problem of determining the degree of nonconstructivity in this acceptance process and argue that since every ZF-sentence will eventually be either accepted or rejected as a true axiom of infinity, the set of all true axioms of infinity therefore exists and (G2) has a definite meaning. Should (G2) hold for this naive notion of true axiom of infinity? If every set is ordinally definable (it may be significant that this notion was first introduced in [1]), then it might be possible for axioms of infinity to determine the full structure of the universe (see [3, §4] for a weak example).

One could perhaps go so far as to compare the (G2) assertion that knowledge about “everything” is reducible to knowledge about “infinity”, and the CH assertion that the class of all hereditarily countable sets can be injected into the class of all hereditarily countable ordinals.

In Theorem 1 of this paper we showed that a weak form of (G2) holds for every model of ZF; and in Theorem 2 we showed that a consequence of Theorem 1 is that in models of ZF, set theory approaches ordinal theory. A viewpoint which we have advocated in [3] is that the process of approximating the width of the universe (e.g. finding new reals) should be as interminable as the process of approximating the length of the universe (e.g. adding more ordinals). It seems that if we wish to emphasize the width of the universe, then we cannot expect any form of (G2) to hold.

The key fact in the proof of Theorem 1 is that in ZF the universe is the union of an increasing ordinal-indexed sequence of sets, the R_α sequence; and this fact is basically a consequence of the power-set axiom. The point we wish to make here is that ZF^- (i.e. ZF set theory without the power-set axiom) is an *essentially richer theory about the class of all sets than ZF*.

In ZF^- one might replace O -sentences by o -sentences, where σ is an o -

sentence iff σ has the form $(\overrightarrow{Q}\beta)(\exists x)[x \models \phi[\overrightarrow{\beta}]]$ for some formula ϕ . It does not seem likely that Theorem 1 can be proved with ZF and OS replaced by ZF^- and oS . Nor does it seem likely that Theorem 2 holds for ZF^- .

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