

THE MEANING OF THE CAUCHY-SCHWARZ-BUNIAKOVSKY INEQUALITY

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ABSTRACT. It is proved that a mapping $T: X \rightarrow X^*$ from a topological real vector space into its dual satisfies the inequality $\langle Ty, x \rangle \leq \langle Tx, x \rangle^{1/2} \cdot \langle Ty, y \rangle^{1/2}$ if and only if it is the restriction of a positively homogeneous subdifferential operator.

The Cauchy-Schwarz-Buniakovsky inequality for a nonnegative definite real quadratic form says

$$(1) \quad \sum a_{ij} x_i y_j \leq \left(\sum a_{ij} x_i x_j \right)^{1/2} \left(\sum a_{ij} y_i y_j \right)^{1/2},$$

where positive roots are taken. Usually an absolute value sign is placed around the left-hand member, to obtain only an apparently stronger inequality. If x denotes the row vector of components x_1, x_2, \dots, x_n , and $x^* = Tx$ the column-vector of components $\sum a_{1j} x_j, \sum a_{2j} x_j, \dots, \sum a_{nj} x_j$, then (1) can be written in the form

$$(2) \quad \langle Ty, x \rangle \leq \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2},$$

the angular brackets indicating the matrix theoretical product of a column-vector and a line-vector. In this form the CSB-inequality appears as a relation satisfied by a mapping $T: X \rightarrow X^*$ from a linear space X into its dual X^* . Therefore, it is natural to ask what such a relation means for the operator T , or, in other words, what can be said of an operator T satisfying the CSB-inequality. In posing such a question nothing should a priori be assumed of T beyond that which is strictly necessary to make the inequality meaningful. Thus, T will just be a mapping, possibly many valued, from X into X^* satisfying the CSB-inequality (2), taken to mean

$$(3) \quad \langle y^*, x \rangle \leq \langle x^*, x \rangle^{1/2} \langle y^*, y \rangle^{1/2},$$

for any x and y in $D(T)$ and any choice of x^* and y^* in Tx and Ty respectively, and where the square root is construed to mean that the quantity under the square root sign as well as the result of the operation is a nonnegative number. Let us recall in passing that the domain of definition of

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a many-valued (or set-valued) mapping T is the set $D(T) = \{x | Tx \neq \emptyset\}$. Easily constructed examples of nonlinear many-valued operators satisfying (2) indicate that none of this generality is superfluous (see [6, Theorem 9.1]).

Our purpose here is to point out that the CSB-inequality does not lead to a new category of objects but to a known class of operators, the positively homogeneous subdifferential operators, and in consequence, that the CSB-inequality sits squarely in the midst of convexity. So, surprisingly, convexity rather than linearity (or "quadraticity") is the context in which the CSB-inequality should be placed. Particular instances of this situation are known. Besides the linear positive selfadjoint operators a conspicuous case is offered by the "duality mapping" of a Banach space into its dual, defined by $Jx = \{x \in X^* | \langle x^*, x \rangle = \|x^*\|^2 = \|x\|^2\}$. This operator, of frequent appearance in convexity and monotonicity theory, satisfies the CSB-inequality and is the subdifferential of half the norm squared [1], [2]. Inequalities formally identical to (2), often going under the same name but of quite a different nature, are obtained by assigning to the angular brackets meanings other than that of the bilinear form effecting the pairing of two dual spaces. Our results, of course, do not apply to them. For instance, such is the case for the CSB-inequality appearing in R. Tapia's papers on the characterization of inner product spaces [4], [5], where the operator is the identity mapping and the brackets denote a sort of generalization (nonbilinear) of the inner product.

We begin by describing the terms and concepts needed for our discussion.

Let X be a locally convex, Hausdorff, topological, real vector space, and let X^* be its dual (i.e. the set of all continuous linear functionals endowed with a topology compatible with the duality between X and X^*). Given in X an everywhere defined, lower semicontinuous, convex function $k(x)$, with values on the extended real line $(-\infty, +\infty]$, we say that a vector $u^* \in X^*$ is a subgradient of k at x , if

$$(4) \quad k(y) - k(x) \geq \langle u^*, y - x \rangle, \quad \forall y \in X.$$

When not empty the set of subgradients of k at x is a closed convex set in X^* ; it is called the "subdifferential" of k at x , and is denoted $\partial k(x)$. The mapping from X into X^* : $x \rightarrow \partial k(x)$, is "the subdifferential operator" associated to k . Subdifferential operators are a subclass of the so-called "cyclically monotone operators". There are mappings $T: X \rightarrow X^*$ satisfying the inequalities

$$(5) \quad \begin{aligned} & \langle x_0^*, x_1 - x_0 \rangle + \langle x_1^*, x_2 - x_1 \rangle + \cdots + \langle x_{n-1}^*, x_n - x_{n-1} \rangle \\ & + \langle x_n^*, x_0 - x_n \rangle \leq 0, \\ & \forall n > 0, \forall x_0, x_1, \dots, x_n \in D(T), \forall x_i^* \in Tx_i, i = 0, 1, \dots, n. \end{aligned}$$

If X is a Banach space the subdifferentials coincide with the maximal cyclically monotone mapping, that is, with cyclically monotone operators admitting no proper cyclically monotone extensions (cf. [3, Theorem 3']). In other words, since any cyclically monotone operator can be extended to a maximal one, a mapping $T: X \rightarrow X^*$ of a Banach space into its dual is cyclically monotone if and only if there is a lower semicontinuous convex function $k(x)$ such that $Tx \subset \partial k(x)$, $\forall x \in D(T)$.

An operator T is said to be positively homogeneous if $T(tx) = tTx$ for $t \geq 0$. As we have already mentioned, it is the class of positively homogeneous subdifferential mappings that is of significance in relation to the CSB-inequality, for the CSB-inequality plays for them the role that (5) played for subdifferential mappings in general. The precise result is the following

THEOREM. *A mapping $T: X \rightarrow X^*$ from a locally convex topological real vector space into its dual satisfies the CSB-inequality if and only if it is the restriction of a positively homogeneous subdifferential operator. Or equivalently, the CSB-inequality holds if and only if there is a lower semicontinuous convex function $k(x)$, positively homogeneous of degree two, such that*

$$(6) \quad Tx \subset \partial k(x), \quad \forall x \in D(T).$$

In such a case $k(x) = \frac{1}{2}\langle x^*, x \rangle$, $\forall x \in D(T)$, $\forall x^* \in Tx$.

PROOF. Assume first that T satisfies the CSB-inequality, and define the real valued function

$$(7) \quad k(x) = \sup_{y \in D(T), y^* \in Ty, t \geq 0} \{t\langle y^*, x \rangle - (t^2/2)\langle y^*, y \rangle\}.$$

As the supremum of a class of affine functions $k(x)$ is a lower semicontinuous convex function; it is also apparent that $k(x)$ is positively homogeneous of degree two. If $x \in D(T)$ and $x^* \in Tx$, then the triplet $x, x^*, t = 1$, is a possible choice competing for the supremum in the definition of k , and hence $k(x) \geq \frac{1}{2}\langle x^*, x \rangle$. On the other hand

$$\frac{1}{2}\langle x^*, x \rangle - k(x) = \inf_{y \in D(T), y^* \in Ty, t \geq 0} \left\{ \frac{1}{2}\langle x^*, x \rangle + (t^2/2)\langle y^*, y \rangle - t\langle y^*, x \rangle \right\},$$

and since the geometric mean is not larger than the arithmetic one,

$$\frac{1}{2}\langle x^*, x \rangle - k(x) \geq \inf_{y \in D(T), y^* \in Ty, t \geq 0} \{t\langle x^*, x \rangle^{1/2}\langle y^*, y \rangle^{1/2} - t\langle y^*, x \rangle\} \geq 0,$$

by the CSB-inequality. Therefore $k(x) = \frac{1}{2}\langle x^*, x \rangle$, $\forall x \in D(T)$, $\forall x^* \in Tx$. In particular, $\frac{1}{2}\langle x^*, x \rangle$ does not depend on the choice of x^* in Tx , and without ambiguity we can write $k(x) = \frac{1}{2}\langle Tx, x \rangle$ for $x \in D(T)$. Next we see that any $x^* \in Tx$ is a subgradient of k . To this end we must show that

$$(8) \quad k(y) - k(x) \geq \langle x^*, y - x \rangle, \quad \forall y \in X,$$

that is, in view of what has already been proved, that

$$\sup \{t\langle z^*, y \rangle - (t^2/2)\langle z^*, z \rangle - \frac{1}{2}\langle x^*, x \rangle - \langle x^*, y - x \rangle\} \geq 0,$$

$$z \in X, z^* \in Tz, t \geq 0,$$

which is obvious since the expression in braces vanishes for $z = x$, $z^* = x^*$, $t = 1$. Thus, $Tx \subset \partial k(x)$, $\forall x \in D(T)$.

We must also verify that $\partial k(x)$ is positively homogeneous of degree one. If $x^* \in \partial k(x)$, then $k(y) - k(x) \geq \langle x^*, y - x \rangle$, $\forall y \in X$, whence replacing y by $t^{-1}y$, $t > 0$, and multiplying by t^2 ,

$$k(y) - k(tx) \geq \langle tx^*, y - tx \rangle, \quad \forall y \in X,$$

that is, $tx^* \in \partial k(tx)$, and so, $t\partial k(x) \subset \partial k(tx)$, $\forall x \in X, t > 0$. This relation is reversed by multiplying by t^{-1} and replacing t and x by t^{-1} and tx respectively, so $\partial k(tx) = t\partial k(x)$, $t > 0$. Thus we have completed the proof that the CSB-inequality is sufficient for T to be the restriction of a positively homogeneous subdifferential operator, as well as the last part of the theorem. To establish necessity we show that any positively homogeneous subdifferential mapping T satisfies the CSB-inequality. This we do by means of inequalities (5), which T , as a subdifferential mapping, satisfies. We take any two points x and y in $D(T)$ and any two of their respective images x^* and y^* , and for any positive integer m set

$$x_i = \begin{cases} (i+1)x, & 0 \leq i \leq m-1, \\ (2m-i)y, & m \leq i \leq 2m-1, \end{cases}$$

$$x_i^* = \begin{cases} (i+1)x^*, & 0 \leq i \leq m-1, \\ (2m-i)y^*, & m \leq i \leq 2m-1. \end{cases}$$

Clearly, $x_i^* \in Tx_i$ by the positive homogeneity of T . With this choice of points (5) yields

$$\begin{aligned} & \langle x^* + 2x^* + \cdots + (m-1)x^*, x \rangle + \langle mx^*, my - mx \rangle \\ & - \langle my^* + (m-1)y^* + \cdots + 2y^*, y \rangle + \langle y^*, x - y \rangle \leq 0, \end{aligned}$$

that is,

$$\begin{aligned} & \frac{1}{2}m(m-1)\langle x^*, x \rangle + m^2\langle x^*, y - x \rangle - \frac{1}{2}(m+2)(m-1)\langle y^*, y \rangle \\ & + \langle y^*, x - y \rangle \leq 0, \end{aligned}$$

whence dividing by m^2 and letting $m \rightarrow \infty$,

$$0 \geq \frac{1}{2}\langle x^*, x \rangle + \langle x^*, y - x \rangle - \frac{1}{2}\langle y^*, y \rangle = \langle x^*, y \rangle - \frac{1}{2}\langle x^*, x \rangle - \frac{1}{2}\langle y^*, y \rangle.$$

By writing ty and ty^* in place of y and y^* , as we may, and letting t approach zero we obtain $\langle x^*, x \rangle \geq 0$, and similarly $\langle y^*, y \rangle \geq 0$. Finally the substitution $x \rightarrow t^{-1}x$, $y \rightarrow ty$, $x^* \rightarrow t^{-1}x^*$, $y^* \rightarrow ty^*$, yields

$$0 \geq \langle x^*, y \rangle - \frac{1}{2}t^{-2}\langle x^*, x \rangle - \frac{1}{2}t^2\langle y^*, y \rangle,$$

from which the CSB-inequality follows by observing that the supremum of the right-hand side is $\langle x^*, y \rangle - \langle x^*, x \rangle^{1/2}\langle y^*, y \rangle^{1/2}$. This brings the proof to an end.

REMARK. If the notation $Tx \subset \partial \frac{1}{2}\langle Tx, x \rangle$ is allowed to mean that there is a l.s.c. convex function $k(x)$ such that $k(x) = \frac{1}{2}\langle Tx, x \rangle$ and $Tx \subset \partial k(x)$, $\forall x \in D(T)$, the conclusion may be drawn that the CSB-inequality is equivalent to the subdifferential relation $Tx \subset \partial \frac{1}{2}\langle Tx, x \rangle$.

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