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SIMPLE PROOF THAT A p-ADIC PASCAL'S TRIANGLE IS 120° ROTATABLE

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ABSTRACT. An array of numbers which is obtained by the replacement of binomial or multinomial coefficients by their p-adic valuation is not only parallel translatable but also 120° rotatable, without changing its configuration in the Pascal's triangle.

Let p be a prime number. *The p-adic Pascal's triangles*, or *Pascal's pyramids* are obtained by the replacement of each binomial or multinomial coefficient by its p-adic valuations. A p-adic Pascal's triangle is *translatable* in the sense that any finite subconfiguration has exact replicas at infinitely many other places in the same array of numbers. Analogous properties with respect to the reflections and rotations of its subconfigurations seem to be less well known. We give a simple proof of the following

THEOREM. *The finite configuration, obtained by* 120° *rotation of any part of a p-adic Pascal's triangle, has exact replicas at infinitely many places in the original p-adic Pascal's triangle.*

PROOF. Consider a binomial coefficient

$$\binom{p^h - 1}{n}$$

and any multinomial coefficient

$$\binom{n}{k_1, k_2, \ldots, k_l}$$

with $0 \leqslant n = k_1 + k_2 + \cdots + k_l \leqslant p^h - 1$. Since

$$\left|\binom{p^h - 1}{n}\right|_p = 1 \text{ for all } 0 \leqslant n \leqslant p^h - 1,$$

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we get an equality

$$\left|\binom{n}{k_1, k_2, \ldots, k_l}\right|_p = \left|\binom{n}{k_1, k_2, \ldots, k_l}\right|_p \cdot \left|\binom{p^h - 1}{n}\right|_p$$
$$= \left|\binom{p^h - 1}{k_1, k_2, \ldots, k_l, p^h - 1 - n}\right|_p.$$

The right side of the equality is invariant under any permutation of $(k_1, k_2, \ldots, k_l, p^h - 1 - n)$. This means that the equilateral triangular, or tetrahedral portion at the top of the p-adic Pascal's triangle, remains unchanged by its automorphic reflections and rotations. Since h is arbitrary, the Theorem is proved. The translatability of configurations also follows immediately, but 60° rotation, is in general, not possible.

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