PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

# THE NONEXISTENCE OF CERTAIN INVARIANT MEASURES

PAUL ERDÖS AND R. DANIEL MAULDIN

ABSTRACT. It is shown that there does not exist an uncountable group $G$ and a nontrivial, $\sigma$-finite, countably additive measure defined on all subsets of $G$ which is left-invariant.

The purpose of this note is to resolve a point left unclear in a recent paper of F. Terpe [1] and its review [2]. In [1], F. Terpe shows that the existence of a certain "maximal" integral is equivalent to the existence of a nontrival countably additive $\sigma$-finite measure $m_I$ defined on all subsets of the interval $I = [0, 1)$ and invariant under translation mod 1. In his review [2] of this paper, J. C. Oxtoby points out that the proof given there for the nonexistence of such a measure tacitly presupposes that the $\sigma$-field $2^I \times 2^I$ of subsets of $I \times I$ generated by generalized rectangles is invariant under the shear map $S$, where $S(x, y) = (x + y, y)$ and addition is mod 1, and that by a theorem of Iwanik [3] this instance of Weil's measurability condition is satisfied if and only if all subsets of $I \times I$ belong to $2^I \times 2^I$. Thus, Terpe's reasoning actually established the nonexistence of $m_I$ only under the hypothesis $2^{I \times I} = 2^I \times 2^I$. Finally, Oxtoby points out in his review that $2^{I\times I} = 2^I \times 2^I$ is implied by CH, but that CH makes the group argument unnecessary. Oxtoby ends his review by stating that the situation is unclear without CH.

We give a short argument below to show that no such hypothesis is needed.

THEOREM. *Suppose $G$ is an uncountable group and $\mu$ is a $\sigma$-finite countably additive left-invariant measure defined on all subsets of $G$. Then $\mu$ is trivial.*

PROOF. Let $M$ be a subgroup of $G$ of cardinality $\aleph_1$. Let $R$ be the family of all right cosets of $M$ and let $A$ be a subset of $G$ which intersects each set in $R$ in exactly one point.

Let $\mathcal{H} = \{mA : m \in M\}$. Then $\mathcal{H}$ is a family of $\aleph_1$ disjoint sets covering $G$ and if $H_1$ and $H_2$ belong to $\mathcal{H}$, then $H_2$ is a left translate of $H_1$.

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of sets of finite measure covering $G$. For each $n$, the sets of the form $K_n \cap H$, where $H \in \mathcal{H}$ form a decomposition of $K_n$ and therefore there are not uncountably many $H$'s with $\mu(K_n \cap H) > 0$.

Thus, there is a set $H_0$ in $\mathcal{H}$ with $\mu(K_n \cap H_0) = 0$ for each $n$. Therefore, $\mu(H) = 0$ for all $H \in \mathcal{H}$. This implies that $\aleph_1$ is a real-valued measurable

Received by the editors September 30, 1975 and, in revised form, January 30, 1976.
*AMS (MOS) subject classifications* (1970). Primary 28A25, 28A70; Secondary 04A10.
*Key words and phrases*. Left-invariant measure, measurable cardinal.

cardinal. But, assuming the axiom of choice (which we are in this paper), it is known that $\aleph_1$ is not measurable [4].

## References

1. Frank Terpe, *Zur Existenz maximaler translationsaleicher Integrale*, Theory of Sets and Topology (in honor of Felix Hausdorff, 1868–1942), VEB Deutscher Verlag, Berlin, 1972, pp. 495–502. MR **49** #10861.

2. J. C. Oxtoby, Math Reviews **49** #10861.

3. A. Iwanik, *On infinite complete algebras*, Colloq. Math. **29** (1974), 195–199, 307. MR **49** #179.

4. S. M. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1930), 141–150.

Department of Mathematics, University of Florida, Gainesville, Florida 32611