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A COMMUTATIVITY THEOREM FOR RINGS

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ABSTRACT. Let R be any associative ring. Suppose that for every pair $(a_1, a_2) \in R \times R$ there exists a pair (p_1, p_2) such that the elements $a_i - a_i^2 p_i(a_i)$ commute, where the p_i's are polynomials over the integers with one (central) indeterminate. It is shown here that the nilpotent elements of R form a commutative ideal N, and that the factor ring R/N is commutative. This result is obtained by the use of the concept of *cohypercenter* of a ring R, which concept parallels the hypercenter of a ring.

Introduction. Let R be any associative ring with center Z. Let $p(t)$ be a polynomial over the integers in one indeterminate. In the early 1950's, I. N. Herstein proved that if R is subject to the condition $x - x^2 p_x(x) \in Z$, all $x \in R$, then $R = Z$ [3]. More recently, Herstein has shown a theorem solving a longstanding question, which asserts that if R is subject to the condition $x_1^{n_1} \cdot x_2^{n_2} = x_2^{n_2} \cdot x_1^{n_1}$, all $x_i \in R$ ($n_i \geqslant 1$ depending on the x_i's), then the ideal commutator of R is nil [5]. This theorem applies to the rings R, which are *radical* over a commutative subring A, in the sense that $x^{n(x)} \in A$, all $x \in R$.

One could look at analogous situations with respect to the $x - x^2 p(x)$ theorem cited above. For instance if the ring R is subject to the condition

$$(\mathrm{C}_0) \qquad x - x^2 p(x) \in A$$

(where A is a commutative subring) does it follow, again, that the ideal commutator is, at least, nil? In this paper we prove the following commutativity theorem. *Suppose that for each pair x_1, x_2 there exists a pair of polynomials $p_1(t)$, $p_2(t)$ such that the elements $x_i - x_i^2 p_i(x_i)$ commute* (C). *Then the ideal commutator of R is nil, and the nilpotent elements form a commutative ideal. Thus R satisfies a multilinear identity of degree* 4 (Theorem 3). This result applies obviously to the rings subject to condition (C_0) (since (C_0) is a stronger condition).

Conventions. All polynomials are polynomials over the integers $\mathbf{Z}$ with the indeterminate t. We denote by $\mathfrak{G}$ the set of polynomials $g(t)$ of the form $g = t - t^2 p(t)$. Clearly $\mathfrak{G}$ is a multiplicatively closed subset (under composition). Given the ring R, we denote by Z the center of R, by J the Jacobson

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radical of R, by N the prime radical of R. If X is a subset of R, $C_R(X) = \{a \in R, ax = xa, \text{ all } x \in X\}$. The commutator $x_1x_2 - x_2x_1$ of the pair (x_1, x_2) is denoted by $[x_1, x_2]$. Finally the rings considered here need not be with 1.

1. **Preliminary results.** We collect some of the facts to be used for our main result (Theorem 3). Remarks 1–4 are known and easy. Although Remark 8 is essentially proven in [4], we have given its complete proof for convenience of the reader.

REMARK 1 (JACOBI'S IDENTITY). $[xy, a] = [x, a]y + x[y, a]$. *If then* $[x, a] = 0$ *it follows that* $[xy, a] = x[y, a]$.

REMARK 2. *If I is a right ideal of R,* $I \cdot [C_R(I), R] = 0$.

REMARK 3. *If R is prime, and if* $I \neq 0$, $C_R(I) = Z$.

REMARK 4. *If R is prime, and if I is commutative,* $R = Z$.

Recall that an element x_0 of R such that $x_0 + x_0' + x_0x_0' = 0$ for some x_0' commuting with x_0 is called *quasi-regular*. Then $1 + x_0$ is invertible with inverse $1 + x_0'$ (formally). The quasi-inner automorphism $x \mapsto x + x_0x + xx_0' + x_0xx_0'$ is denoted as usual by $x \mapsto (1 + x_0)x(1 + x_0)^{-1}$.

REMARK 5 [4]. *If x_0 is quasi-regular,*

$$[x_0, a](1 + x_0)^{-1} = a - (1 + x_0)a(1 + x_0)^{-1}.$$

REMARK 6 [4]. *If A is an additive subgroup of the ring R, which is preserved re quasi-inner automorphisms, and if I is a quasi-regular right ideal, then* $A \cap I = 0$ *implies* $A \subseteq C_R(I)$.

REMARK 7 [4]. *If A is as in Remark 6, and if A contains no nilpotent elements, for any* $a \in A$ *with* $au = 0$, *we have* $uRa^2 = 0$. *Thus if R is prime, A has no divisors of zero on R.*

REMARK 8 (BRAUER, HERSTEIN). *Suppose that the sequence* (x, z, b_1, b_2) *in R has the following properties*:

(i) z, b_1, b_2 *commute pairwise*;

(ii) x, $-z \neq 0$, zx *are quasi-regular*;

(iii) *If* $c_i = [x, b_i]$ *is a zero divisor then* $c_i = 0$;

(iv) $[(1 + x)b_2(1 + x)^{-1}, b_1] = 0$, $[(1 + zx)b_2(1 + zx)^{-1}, b_1] = 0$.

Then at least one of the c_i's must be 0.

PROOF. Let $a_1 = (1 + x)b_2(1 + x)^{-1}$ and $a_2 = (1 + zx)b_2(1 + zx)^{-1}$. We have by (iv),

(v) $[a_1, b_1] = [a_2, b_1] = 0$,

and we have

$$(\mathrm{E}_1) \qquad (1 + x)b_2 = a_1(1 + x), \qquad (1 + zx)b_2 = a_2(1 + x).$$

An elementary operation gives

$$(\mathrm{E}_2) \qquad (z - 1)b_2 = za_1 - a_2 + (za_1 - a_2z)x.$$

Now by (i) both b_2 and z commute with b_1, so, $[(z - 1)b_2, b_1] = 0$. Also z,

a_1, a_2 commute all with b_1 ((v)), thus $[za_1 - a_2, b_1] = 0$. Since by Jacobi's identity $[(za_1 - a_2 z)x, b_1] = (za_1 - a_2 z)[x, b_1]$, we see that ($E_2$) implies

$$(E_3) \qquad (za_1 - a_2 z) \cdot [x, b_1] = (za_1 - a_2 z)c_1 = 0.$$

If then $c_1 \neq 0$ is a nonzero divisor, (E_3) gives $za_1 - a_2 z = 0$. Going back to (E_2) we get

$$(E_4) \qquad b_2(z-1) = (z-1)b_2 = za_1 - a_2 = a_2 z - a_2 = a_2(z-1),$$

that is, $(b_2 - a_2)(z-1) = 0$, so, $(z-1)^{-1}(b_2 - a_2)(z-1) = 0$, consequently $b_2 = a_2$. From the second equation in (E_1) follows $[b_2, zx] = 0$. Since $[b_2, z] = 0$ we get $z[b_2, x] = 0$, so, $z \cdot c_2 = 0$. Since c_2 is a nonzero divisor or 0 and since $z \neq 0$, $c_2 = 0$ necessarily, thereby proving Remark 8.

REMARK 9. *Let R be a prime ring having a radical ($J \neq 0$). Any commutative subgroup A of R containing no nilpotent elements, which is preserved re quasi-inner automorphisms, must be contained in the center.*

PROOF. If $A \cap J = 0$, then by Remark 6, $A \subseteq C_R(J)$, and by Remark 3, $C_R(J) = Z$, so $A \subseteq Z$. If, on the other hand, $A \cap J \neq 0$, choose $z \neq 0$ in $A \cap J$, and any $a \in A$, $x \in J$. If $b_1 = b_2 = a$, the sequence (x, z, b_1, b_2) has all the requirements as in Remark 8. Let us show, for example (iii). Let $c = [x, a]$. If $c \neq 0$, then $c(1+x)^{-1} \neq 0$ is in A (Remark 5). By Remark 7, A has no divisors on R, thus $c(1+x)^{-1}$ is a nonzero divisor. It follows that c is a nonzero divisor (or zero). Applying Remark 8 we see that $c = 0$ necessarily, that is, $[x, a] = 0$. Thus $A \subseteq C_R(J) = Z$ as wished.

REMARK 10. *Let $\mathcal{E}_0$ be a closed set of polynomials. Suppose that for every (a_1, a_2) there is $(g_1, g_2) \in \mathcal{E}_0 \times \mathcal{E}_0$ such that $[g_1(a_1), g_2(a_2)] = 0$. Let $a, x \in R$ with the following conditions.*

(i) *$-a$, x, ax are quasi-regular;*

(ii) *$[x, \mathcal{E}_0(a)]$ has no proper divisors of zero on R.*

Then for some $g \in \mathcal{E}_0$, $[x, g(a)] = 0$.

PROOF. Applying (C_0) for $((1+x)a(1+x)^{-1}, a)$ we get for some g_1, $h_1 \in \mathcal{E}_0$,

$$[(1+x)g_1(a)(1+x)^{-1}, h_1(a)] = 0.$$

Applying (C_0) for $[(1+ax)g_1(a)(1+ax)^{-1}, h_1(a)]$ we get for some g_2, $h_2 \in \mathcal{E}_0$,

$$[(1+ax)g_2(g_1(a))(1+ax)^{-1}, h_2(h_1(a))] = 0.$$

Setting $b_1 = h_2(h_1(a))$, $b_2 = g_2(g_1(a))$, $z = a$, we see that (x, z, b_1, b_2) satisfies the hypothesis in Remark 8, so $[x, b_1] = 0$ or $[x, b_2] = 0$, with $b_i = g(a)$, some $g \in \mathcal{E}_0$.

2. **Cohypercenter.** The analog with respect to condition (C) (see Introduction) of Herstein's hypercenter [4] is the following set $T = T(R)$.

DEFINITION. *$a \in T$ if and only if given $x \in R$ there exists $p(t)$ such that $[a, x - x^2p(x)] = 0$, where $p(t)$ is a polynomial having integral coefficients depending on (a, x).*

Let us call T the *cohypercenter* of R. We wish to get the property $T = Z$ for, at least, the class of semiprime rings.

REMARK 11. *Given $a, b \in T$ and $x \in R$, there is $p(t)$ such that $[a, x - x^2p(x)] = [b, x - x^2p(x)] = 0$.*

PROOF. Since $a \in T$ there is $g_1 \in \mathfrak{S}\,(= \{t - t^2\mathbf{Z}[t]\})$ such that $[a, g_1(x)] = 0$. Since $b \in T$ there is $g_2 \in \mathfrak{S}$ such that $[b, g_2(g_1(x))] = 0$. If $g = g_2 \circ g_1$, then $g \in \mathfrak{S}$, and $[a, g(x)] = [b, g(x)] = 0$.

REMARK 12. *T is a commutative subring evidently preserved re quasi-inner automorphisms.*

PROOF. Given $a, b \in T$ and x there is $g \in \mathfrak{S}$ such that $[a, g(x)] = [b, g(x)] = 0$ (Remark 11). Thus $[a \pm b, g(x)] = [ab, g(x)] = 0$, and $x \in C_R(T_0)$, where T_0 is the subring of R generated by a, and b shows that T is a subring and that $x - x^2p_x(x) = g_x(x) \in Z(T_0)$, all $x \in T_0 \subseteq R$. By Herstein [3], T_0 is commutative. Therefore $[a, b] = 0$, all $a, b \in T$, that is, T is commutative.

REMARK 13 [2, LEMMA 2]. *Let V be a space over the division ring D. If f is a linear transformation sending each $\vec{v} \in V$ to $\vec{v}\lambda$, $\lambda \in D$, depending on $\vec{v}$, and if $\dim V > 1$, then f is induced by a scalar (central element λ_0 of D).*

THEOREM 1. *If R is any semiprime ring, the cohypercenter of R is precisely the center of R.*

PROOF. We prove the assertion by a step-by-step reduction from division rings to the considered rings.

First if R is a division ring, it is clear that $T = Z$ (Remark 9, and Brauer, Cartan, Hua result).

Next if R is a primitive ring, which is not a division ring, then by the density theorem, R acts densely on a space V over the division ring D with a dimension > 1. Let $\vec{v} \in V$ and let $a \in T$. If $\vec{v}$ and $\vec{v}a$ were not collinear, by the density action of R, there is $x \in R$ with $\vec{v}x = 0$ and $\vec{v}ax = \vec{v}$. Since $a \in T$, there is $y = g(x) = x - x^2p(x)$ with $[a, y] = 0$, whence, $\vec{v}ay = \vec{v}ya$. Now

$$\vec{v}ay = \vec{v}ax - \vec{v}ax(xp(x)) = \vec{v} - \vec{v}(xp(x)) = \vec{v};$$

but, $\vec{v}ya = \vec{v}x(1 - x(p(x)))a = \vec{0}$, a contradiction. This shows that $\vec{v}a = \vec{v}\lambda$, some $\lambda \in D$. By Remark 13, a is induced by a scalar in D. Therefore $a \in Z(R)$.

Next if R is *semisimple* ($J = 0$), then R is a subdirect product of primitive rings $\overline{R}$. By the above $\overline{T} = T(\overline{R}) = \overline{Z} = Z(\overline{R})$. Since T maps into $\overline{T}$, T is central in $\overline{R}$. Therefore $T = Z$.

Next suppose that R is prime but not semisimple. If $a^2 = 0$ with $a \in T$, we claim that $a = 0$. In fact if $a \neq 0$, $I = aR \neq 0$. Since $a \in T$, for every $x \in R$, there is $g \in \mathfrak{S}$ such that $[a, g(ax)] = 0$. Thus for some $p(t)$, $axa = (ax)^2p(ax)a$. Multiplication on the right by x gives $(ax)^2 = (ax)^3p(ax)$.

Therefore $y^2 = y^3 p_y(y)$, all $y \in I$. Now if I were nil, $y^2 = 0$, all $y \in I$, so, R would contain a nilpotent ideal (Levitski's result), a contradiction. This shows that the π-regular ring I must contain some idempotent $e \neq 0$. If $R_0 = eRe$, this is a nonzero prime ring verifying again $y^2 = y^3 p_y(y)$ (since $R_0 \subseteq eR \subseteq I$). Since by [2], R_0 satisfies a polynomial identity and since R_0 is π-regular, R_0 is the ring of matrices over a division ring. Thus R_0, whence R, have nonzero socles, contrary to the choice of R. This shows that $a = 0$ necessarily, and T contains no nilpotents, By Remarks 12 and 9, $T = Z$ follows.

All in all, we have shown that if R is prime, $T = Z$. We go back to the semiprime ring R. Since R is a subdirect product of prime rings satisfying the desired conclusion, it follows that $T = Z$, thereby proving the theorem.

3. **The main result.**

REMARK 14. *If R satisfies* (C), *any nilpotent element a must belong to T.*

PROOF. Given $a, b \in R$, we claim that for any given integer n_0 there is $n \geqslant n_0$ such that $[a - a^n p_1(a), b - b^2 p_2(b)] = 0$. In fact, by the basic property, $[a - a^2 p_1(a), b - b^2 p_2(b)] = 0$. If $a_1 = a^2 p_1(a)$, $b_1 = b - b^2 p_2(b)$, there are p_{11}, p_{22} such that

$$[a_1 - a_1^2 p_{11}(a), b_1 - b_1^2 p_{22}(b_1)] = 0.$$

Now the first commutator relation gives $[a - a_1, b_1 - b_1^2 p_{22}(b_1)] = 0$. Adding to the preceding, we get for some $p_{21}(t)$, $[a - a^4 \cdot p_{21}(a), b - b^2 p_{22}(b)] = 0$. Continuing in this way, we see that for any n_0, $n = 2^{n_0}$ will do. If then a is nilpotent of index n_0, for any $b \in R$,

$$[a - a^{2^{n_0}} p'_{n_0}(a), b - b^2 p'_2(b)] = [a, b - b^2 p'_2(b)] = 0$$

tell us that $a \in T$.

THEOREM 2. *If R is a semiprime ring satisfying* (C), *then R is commutative.*

PROOF. First suppose that R is a division ring. Let a, $x \in R$. If condition (i) in Remark 10 does not hold, then evidently $[a, x] = 0$. If, on the other hand, (i) holds, by Remark 10, $[x, g(a)] = 0$, some $g \in \mathfrak{S}$. This shows that for any $x \in R$, $x \in T$. By Theorem 1, $R = Z$.

Next suppose that R is primitive. Every homomorphic image $\overline{R}$ of a subring of R inherits condition (C), which in view of Remarks 14 and 12, tells us that in $\overline{R}$ the nilpotent elements commute. Since this is patently false for the matrix rings over division rings of rank > 1, by a routine argument, R must be a division ring, so, by the above, R must be a field.

Having proved the assertion for the primitive rings we derive, as before, that it holds for the semisimple rings.

Next suppose that R is prime but not semisimple. Since R is evidently semiprime, $T = Z$ (Theorem 1). By Remark 14, R has no nilpotent elements. It follows that R has no divisors of zero. Thus the ring J is a ring subject to condition (C), and has no divisors of zero. Let x, $a \in J$. The pair (a,x) satisfies the hypothesis of Remark 10. Consequently for some $g \in \mathfrak{S}$, $[x, g(a)] = 0$. This shows that $x \in T(J)$, all $x \in J$. By Theorem 1, $J = Z(J)$ follows, and by Remark 4, $R = Z$.

Having proved the assertion for the prime rings we derive the desired result.

Let R be any ring subject to condition (C). Since the factor ring R/N inherits (C) and is semiprime, Theorem 2 applies and yields R/N commutative, whence, R/N has no nilpotent elements. This means that N is the set of nilpotent elements of R. By Remark 14, $N \subseteq T$ is also commutative. If then I is the ideal commutator of R, $I \subseteq N$ is commutative, so R satisfies the identity $[[x_1, x_2], [x_3, x_4]] = 0$. Summarizing we get the main result.

THEOREM 3. *Suppose that for each pair x_1, x_2 in the ring R there exists a pair $(p_1(t), p_2(t))$ of polynomials with integral coefficients such that the elements $x_i - x_i^2 p_i(x_i)$ commute. Then the ideal commutator of R is nil, and the nilpotent elements form a commutative ideal.*

One final remark is in order.

REMARK 15 (*A noncommutative ring as in Theorem* 3). We observed earlier that the rings R subject to the condition (C_0) (see Introduction) satisfy (C), so, the conclusion in Theorem 3 holds for such a class. One could wonder if the latter rings are commutative. If R is the ring of matrices $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ over the field F, and if F is algebraic over a finite field, it was verified in [1] that for every $x \in R$, either $x^2 = 0$ or $x = x^{n(x)+1}$, $n(x) \geqslant 1$. Thus R satisfies condition (C_0) re the commutative subring, even commutative ideal, $A = J = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Here $N = J$ is the set of nilpotent elements, and $R/N \approx F \times F$ is in fact commutative. Is this example the most general one for the noncommutative subdirectly irreducible rings R subject to (C)? If true this would give that the ideal commutator of any ring R with (C) is a square-zero ideal.

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