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GROUP RINGS WITH SOLVABLE n-ENGEL UNIT GROUPS[1]

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ABSTRACT. Let KG be the group ring of a group G over a field of characteristic $p \geqslant 0, p \neq 2, 3$. Suppose G contains no element of order p (if $p > 0$). Group algebras KG with unit group $U(KG)$ solvable and n-Engel are characterized.

Let KG be the group ring of a group G over a field K of characteristic $p \geqslant 0$ and let $U(KG)$ denote its group of units. Several authors including Bateman [1], Bateman and Coleman [2], Motose and Tominaga [10] and Khripta [5] have studied the question as to when $U(KG)$ is solvable or nilpotent. Khripta in a beautiful paper [5] has proved that if $p > 0$ and G has a p-element then $U(KG)$ is nilpotent if and only if G is nilpotent and the derived group G' is a finite p-group, settling the nonsemiprime case. This, incidently, is equivalent to saying that KG is Lie nilpotent (see [11] and [14]). Khripta also has some results in her thesis on the nilpotency of $U(KG)$ in the semiprime case. We investigate when $U(KG)$ is a solvable n-Engel group; more precisely we prove

THEOREM. *Suppose KG is a group ring over a field K of characteristic $p \geqslant 0$, $p \neq 2, 3$. Suppose G has no element of order p (if $p > 0$). Then the following are equivalent.*

(i) *$U(KG)$ is solvable and n-Engel.*

(ii) *G is solvable and m-Engel and one of* (a), (b) *holds.*

(a) *$T(G)$, the set of torsion elements of G, is central in G.*

(b) *$|K| = 2^{\beta} - 1 = p$, a Mersenne prime; $T(G)$ is abelian of exponent $(p^2 - 1)$ and for $x \in G$, $t \in T(G)$, $xt \neq tx \Rightarrow x^{-1}tx = t^p$.*

(iii) *$U(KG)$ is nilpotent.*

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1. **Notations and definitions**. For group elements x, y we write the commutator $(x, y) = xyx^{-1}y^{-1}$ and

$$\Big(x, \underbrace{y, y, \ldots, y}_{n+1}\Big) = \Big(x, \underbrace{y, \ldots, y}_{n}\Big) y \Big(x, \underbrace{y, \ldots, y}_{n}\Big)^{-1} y^{-1}.$$

A group H is n-Engel if it satisfies

$$\Big(x, \underbrace{y, \ldots, y}_{n}\Big) = 1 \quad \text{for all } x, y \in H$$

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and fixed n. Let $\dot{F}$ be the multiplicative group of a field F. We denote by $\mathfrak{E} = \mathfrak{E}(\dot{F})$, the ring of endomorphisms of $\dot{F}$. We write f^α for the image of f under α for $f \in \dot{F}$, $\alpha \in \mathfrak{E}$. Thus $f^{\alpha+\beta} = f^\alpha \cdot f^\beta$ and $f^{\alpha\beta} = (f^\alpha)^\beta$.

By a crossed product $K(G, \rho_{g,h}, \alpha_g)$, we understand the set of finite sums, $\{\sum k_i \bar{g}_i | k_i \in K, g_i \in G\}$ where $\bar{g}_i$ is a symbol corresponding to g_i and ρ: $G \times G \to \dot{K}$ is a factor system and α_g is an automorphism of K for each $g \in G$. Equality and addition are defined componentwise. And, for $g, h \in G$, $k \in \dot{K}$, $\bar{g} \cdot \bar{h} = \rho_{g,h}\overline{gh}$, $gk = k^{\alpha_g} g$ where ρ and α are required to satisfy the necessary conditions for $K(G, \rho_{g,h}, \alpha_g)$ to be a ring. For details, we refer to [3].

As a special case, if we have $\alpha_g = I$ for all $g \in G$, we call

$$K(G, \rho_{g,h}, I) = K^t(G)$$

the twisted group ring (see [12]). If

$$\rho_{g,h} = 1 \quad \text{for all } g, h \in G,$$

we call $K(G, 1, \alpha_g)$, the skew group ring and denote it by $K_\alpha(G)$. And, of course, if also $\alpha_g = I$ for all $g \in G$, we have the (ordinary) group ring. We shall have occasion to use both skew and twisted group rings.

2. **The skew group ring of an infinite cyclic group.** Let F be a field contained in KG. Suppose that $x \in G$ has infinite order, $\langle x \rangle$ is linearly independent over F, and that x induces an automorphism $\alpha = \alpha_x$ of F by conjugation, i.e. $\alpha: f \to xfx^{-1} = f^\alpha$. Then we have an isomorphic copy of the skew group ring $F_\alpha\langle x \rangle$ contained in KG. Hence $F_\alpha\langle x \rangle = \{\sum f_i x^i | f_i \in F\}$ where addition and equality are componentwise and $xf = f^\alpha x$. We investigate $F_\alpha\langle x \rangle$ in this section.

Lemma 2.1. *For all $f \in \dot{F}$, we have*

$$(f, \underbrace{x, x, \ldots, x}_{m}) = f^{(1-\alpha)^m}. \tag{2.2}$$

Proof. We use induction on m. Notice that

$$(f, x) = fxf^{-1}x^{-1} = f \cdot (f^{-1})^\alpha = f \cdot f^{-\alpha} = f^{(1-\alpha)}.$$

Suppose we already know that (2.2) holds for m; then

$$\begin{aligned}(f, \underbrace{x, x, \ldots, x}_{m+1}) &= f^{(1-\alpha)^m} x \left(f^{(1-\alpha)^m}\right)^{-1} x^{-1} \\ &= f^{(1-\alpha)^m} x f^{-(1-\alpha)^m} x^{-1} = f^{(1-\alpha)^{m+1}}.\end{aligned}$$

The lemma is proved.

Proposition 2.3. *Let F be an infinite field of characteristic $p \geqslant 0$ and α be an automorphism of finite order. Suppose that in $F_\alpha\langle x \rangle$ we have*

$$(f, \underbrace{x, x, \ldots, x}_{m}) = 1 \quad \textit{for all } f \in \dot{F}.$$

Then $F_\alpha\langle x \rangle = F\langle x \rangle$, i.e. α is the identity automorphism.

Proof. We have by the last lemma, for all $f \in \dot{F}$,

$$1 = f^{(1-\alpha)^m} = f^{\Sigma_0^m (-1)^i \binom{m}{i} \alpha^i}.$$

Let $s > 1$ be the order of α. Choose a prime $q > \max(p, m)$ of the form $2sk + 1$. Then

$$1 = f^{(1-\alpha)^q} = f^{\Sigma \binom{q}{i} (-1)^i \alpha^i}.$$

The finite automorphism group $I, \alpha, \alpha^2, \ldots, \alpha^{s-1}$ satisfies

$$h\big(f, \alpha(f), \ldots, \alpha^{s-1}(f)\big) = 0$$

with

$$\begin{aligned} h(X_0, X_1, \ldots, X_{s-1}) & \\ = X_0\big(X_0^a \cdot X_{i_1}^{a_1} \cdot X_{i_2}^{a_2} \cdots X_{i_r}^{a_r} - X_{j_1}^{b_1} \cdot X_{j_2}^{b_2} \cdots X_{j_t}^{b_t}\big), & \quad (2.4) \\ r + t = s - 1, & \end{aligned}$$

and $a = |\Sigma_{s=0}^{2k} (-1)^{js} \binom{q}{js}|$. Since $a \equiv 1 \bmod q$, (2.4) is a nontrivial polynomial, contradicting Artin's theorem on the algebraic independence of automorphisms of an infinite field [6, p. 228]. Hence $s = 1$ and $\alpha = I$.

PROPOSITION 2.5. *Let F be a finite field of p^a elements. If $F_\alpha\langle x\rangle$ satisfies*

$$(f, \underbrace{x, x, \ldots, x}_{m}) = 1 \quad \textit{for all } f \in \dot{F},$$

and α is not the identity automorphism; then, $f^\alpha = f^p$ for all $f \in \dot{F}$ and $|F| = p^2$, where p is a Mersenne prime.

PROOF. Since $f^\alpha = f^{p^j}$ for some $j < a$, we have that

$$(f, \underbrace{x, x, \ldots, x}_{m}) = f^{(1-\alpha)^m} = f^{(1-p^j)^m} = 1 \quad \text{for all } f \in \dot{F}.$$

Therefore, $(p^a - 1)$ divides $(p^j - 1)^m$. Hence,

(2.6) any prime divisor of $(p^a - 1)$ divides $(p^j - 1)$.

We claim that (2.6) implies $a = 2$. Let j be the smallest natural number such that (2.6) holds for a fixed a. Then writing, $a = jq + r$,

$$p^a - 1 = p^{jq+r} - 1 = p^r(p^{jq} - 1) + (p^r - 1),$$

it follows that any prime divisor of $(p^a - 1)$ is a divisor of $(p^r - 1)$. We may thus assume that $a = jq$. We have now that any prime divisor of $(p^j)^q - 1$ is a divisor of $(p^j - 1)$. It is easy to see (cf. [9]) that $q = 2$ and $p^j = 2^\gamma - 1$. It follows by [15, p. 335] that $j = 1$. Thus $a = 2$. We have therefore proved that $|F| = p^2, p = 2^\gamma - 1$ and hence $f^\alpha = f^p$.

3. **Proof of the theorem**. We need the following crucial result of Lanski.

THEOREM 3.1 (LANSKI). *Let R be a semiprime ring which is 6-torsion free. If $U(R)$ is solvable, then all idempotents of R are central.*

PROOF. See [7, Lemma 5] and [8, Theorem 9 and §1].

We shall prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

3.2 (i) $\Rightarrow$ (ii): Let g and h be elements of finite order of G. Since

$$e = (1/O(g)) \sum_{1}^{O(g)} g^i$$

is an idempotent, $\langle g \rangle$ is normal by (3.1). Also $\langle h \rangle$ is normal. Thus $T_0 = \langle g, h \rangle$ is a finite normal subgroup of G. Now,

$$KT_0 = \sum_{i}^{\oplus} (D_i)_{n_i},$$

a direct sum of full matrix rings $(D_i)_{n_i}$ over division rings D_i. It follows by [4] that each $n_i = 1$ and each D_i is a commutative field F_i. Hence $gh = hg$. Thus $T = T(G)$, the torsion elements of G form a normal abelian subgroup of G.

Let $x \in G$, $x \notin T$ and let T_0 be a finite subgroup of T. Suppose that x does not commute with T_0 elementwise. Since every finite subgroup is normal in G, the skew group ring $(KT_0)_\alpha\langle x \rangle$ is contained in KG, where α is the automorphism of KT_0 induced by conjugation by x. Now, $KT_0 = \Sigma^{\oplus} F_i$, where F_i are fields. Also,

$$KG \supset (KT_0)_\alpha\langle x \rangle = \left(\sum^{\oplus} F_i \right)_\alpha \langle x \rangle \simeq \sum^{\oplus} (F_i)_\alpha \langle x \rangle. \tag{3.3}$$

The last isomorphism follows because every idempotent is central in KG by (3.1) and $xF_ix^{-1} = F_i$.

We can conclude from (3.3) that the unit group of each $(F_i)_\alpha\langle x \rangle$ is n-Engel. Since F_i is algebraic over K, it follows by Propositions 2.3 and 2.5 that $|K| = p$ or p^2, where p is a Mersenne prime. If $|K| = p^2$ then

$$|F_i| = |K| \Rightarrow F_i = e(KT_0) = eK, \qquad e^2 = e.$$

Since every idempotent is central, F_i commutes with x. Thus we have $|K| = p = 2^\beta - 1$. It remains to prove that $T_0^{(p^2-1)} = 1$ and

$$xt \neq tx, \qquad t \in T_0 \Rightarrow x^{-1}tx = t^p.$$

We first make two observations. Write $T_0 = E \times A$, where E is a 2-group and A is an odd group.

3.4. A is central.

Let $g \in A$, then since x^2 is central, $\langle x, g \rangle / \langle x^2 \rangle$ is a nilpotent group of order $2 \cdot O(g)$. Thus $xgx^{-1} = gx^{2l}$ and also $xgx^{-1} = g^i$ as $\langle g \rangle$ is normal in G. Hence $xgx^{-1} = g$.

3.5. If g and h are nonidentity elements of T_0 then $(1 - g)(1 - h) \neq 0$. This is because the coefficient of identity in this product is 1 or 2 and $p \neq 2$.

Suppose that $T_0^{(p^2-1)} \neq 1$. Choose g, $h \in T_0$ with $h^x h^{-1} \neq 1$ and $g^{p^2-1} \neq 1$. Then

$$\pi = (1 - g^{p^2-1})(1 - h^x h^{-1}) \neq 0.$$

Therefore, there exists an F_i and a homomorphism

$$\lambda\colon KT_0 \to F_i \tag{3.6}$$

with $\lambda(\pi) \neq 0$. Thus $\lambda(g)^{p^2-1} \neq 1$ and $|F_i| > p^2$. Since $\lambda(h^x h^{-1}) \neq 1$, we have $\lambda(h^x) = \lambda(h)^x \neq \lambda(h)$ and F_i is not central, contradicting Proposition 2.5. We have therefore proved that $T_0^{(p^2-1)} = 1$.

In order to complete the proof of the implication (i) $\Rightarrow$ (ii) it suffices to prove

$$(3.7) \qquad g \in T_0, \qquad xg \neq gx \Rightarrow x^{-1}gx = g^p.$$

We can write $g = g_1 g_2$, $O(g_1) = 2^s$ and $O(g_2)$ a divisor of $(p-1)/2$. Since $g_2^p = g_2$ and g_2 is central due to (3.4), we have only to prove that $x^{-1}g_1x = g_1^p$.

We may assume that $s > 1$. Suppose that $K\langle g_1\rangle = F_1 \oplus F_2 \oplus \cdots$, $|F_1| = p^2 = |F_2|$ and $g_1 = (\xi, \eta, \ldots)$, $x^{-1}g_1x = (\xi^p, \eta, \ldots)$. Since $x^{-1}g_1x = g_1^i$, we have $p - i \equiv 0 \pmod 4$ and $i - 1 \equiv 0 \pmod 4$ and thus $p - 1 \equiv 0 \pmod 4$ which is a contradiction. Hence $x^{-1}gx = g^p$.

3.8. (ii) $\Rightarrow$ (iii).

3.9. We assert that every idempotent of KT is central in KG. If (ii)(a) holds, the assertion is trivial. So let us assume (ii)(b). Let $e = e^2 = \Sigma e_g g$. Then $e = e^p = \Sigma e_g g^p$ and therefore $e_g = e_{g^p}$. Now $e^x = \Sigma e_g g^x = e$, since $g^x = g$ or g^p.

Since G is m-Engel solvable it follows by [13, Theorem 7.36] that $G/T(G)$ is nilpotent (say of class $\leqslant c$). We have that either $T(G)$ is central or $|K| = p = 2^\beta - 1$ satisfying (ii)(b). We shall prove that $U(KG)$ is nilpotent of class $\leqslant (c + \beta + 1)$. We may therefore assume that G is finitely generated and, hence, by [13, Theorem 7.34] that G is nilpotent. Therefore $T = T(G)$ is finite.

We have, $KT = \Sigma^{\oplus} F_i$ a finite direct sum of fields. Due to (3.9),

$$KG = (KT)(G/T, \rho, \alpha) = \sum^{\oplus} F_i(G/T, \rho, \alpha).$$

Since G/T is ordered, $U(KG) = \prod^{\otimes} \dot{F}_i \cdot G/T$. It suffices to prove that $\dot{F}_i \cdot G/T$ is nilpotent of class $\leqslant c + \beta + 1$. This is clear if α is trivial, i.e. if F_i and G/T commute. We may therefore suppose that we have $|F_i| = p^2$, $p = 2^\beta - 1$ and we wish to prove that $F_i \cdot G/T$ is nilpotent of class $\leqslant c + \beta + 1$. It is easy to see that $F_i \subset z_{\beta+1}$, the $(\beta + 1)$th term of the upper central series of $(F_i \cdot G/T)$. Since G/T is nilpotent of class $\leqslant c$, $F_i \cdot G/T$ is nilpotent of class $\leqslant (c + \beta + 1)$.

3.10. (iii) $\Rightarrow$ (i) is trivial.

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