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A MULTIPLIER THEOREM FOR SU(n)

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ABSTRACT. Let $G = \mathrm{SU}(n)$, let $\mathfrak{g}$ be its Lie algebra and let m be a function on $\mathfrak{g}$, invariant under the adjoint action of G, which is continuous at the points of $\hat{G}$ (which can be imbedded in $\mathfrak{g}$). If $1 \leqslant p < 2[1 - (n + 2)^{-1}]$ and m is a multiplier for the Ad_G-invariant L^p functions on $\mathfrak{g}$, then the restriction of a translate of m to $\hat{G}$ is a multiplier for the central L^p functions on G.

1. **Introduction**. A function m on the space $\hat{G}$ of (equivalence classes of) irreducible unitary representations of a group G is said to be an $L^p(G)$ multiplier if the transformation T_m defined on a suitable dense subspace of $L^p(G)$ by $(T_m f)^\wedge = m\hat{f}$ can be extended to a bounded operator on all of $L^p(G)$. In this case, $N(m;\, \mathfrak{M}^p(G))$ denotes the operator norm of T_m.

A well-known result of de Leeuw [3] states that if m is an $L^p(\mathbf{R})$ multiplier which is continuous at the integers, then the restriction of m to $\mathbf{Z}$ ($= \hat{T}$) is an $L^p(T)$ multiplier, where T is the circle group. Recent work by several authors [2], [4], [5], [6] leads naturally to the conjecture that the de Leeuw result remains true if T is replaced by any compact Lie group G, $\mathbf{R}$ by the Lie algebra $\mathfrak{g}$ of G regarded as a vector group, and $\mathbf{Z}$ by $\hat{G}$, which can be naturally imbedded in $\mathfrak{g}$.

In this note, we extend a partial result in the direction of the conjecture proved in [5] by Strichartz for $G = \mathrm{SO}(3)$ to the case $G = \mathrm{SU}(n)$ or a group covered by $\mathrm{SU}(n)$.

We are concerned here with invariant functions on $\mathfrak{g}$ and G, where invariance on $\mathfrak{g}$ is under the adjoint action of G and invariance on G is under conjugation, so that invariant functions on G are simply central functions. The notation L_i^p will be used to denote L^p spaces of invariant functions. Roots and weights on $\mathfrak{g}$, and thus representations $\lambda \in \hat{G}$ can be regarded as members of $\mathfrak{g}$ via the Killing form, which is denoted by $\langle\ ,\ \rangle$; β is one-half the sum of the positive roots.

THEOREM. *Suppose that $G = \mathrm{SU}(n)$ and that m is an invariant function on $\mathfrak{g}$ which is continuous at $\lambda + \beta$ for all $\lambda \in \hat{G}$. Let $\tilde{m}$ be defined on $\hat{G}$ by $\tilde{m}(\lambda) = m(\lambda + \beta)$. Suppose further that $1 \leqslant p < 2[1 - (n + 2)^{-1}]$ and that m is an $L_i^p(\mathfrak{g})$ multiplier.*

Then $\tilde{m}$ is an $L_i^p(G)$ multiplier and

$$N(\tilde{m};\, \mathfrak{M}_i^p(G)) \leqslant C N(m;\, \mathfrak{M}_i^p(\mathfrak{g})).$$

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(Here and below, unidentified constants depend on at most G and p and may take on different values at different appearances.)

2. **Preliminaries**. We set down here the facts that we need from [5]; for standard facts about compact Lie groups and SU(n) in particular, the reader is referred to [1].

For the moment, G is an arbitrary semisimple compact Lie group with maximal torus H having the Lie algebra $\mathfrak{h}$. Positive roots $\alpha_1, \ldots, \alpha_r$ are chosen in $\mathfrak{h}$ and we set

$$P(X) = \prod_{j=1}^{r} \langle X, \alpha_j \rangle, \quad \Delta(X) = 2i \prod_{j=1}^{r} \sin \frac{1}{2} \langle X, \alpha_j \rangle, \quad \Delta_0(X) = \frac{\Delta(X)}{P(X)}$$

for $X \in \mathfrak{h}$ and extend the domain of Δ_0 to all of $\mathfrak{g}$ by Ad_G invariance.

The main tool enabling us to relate Fourier analysis on G to Fourier analysis on $\mathfrak{g}$ is the map $F \to \tilde{F}$ from $L_i^1(\mathfrak{g})$ to $L_i^1(G)$ defined by

$$\tilde{F}(\exp X) = \sum_{\mathfrak{L}} F(X + l)\Delta_0(X + l)^{-1}, \qquad X \in \mathfrak{h},$$

where $\mathfrak{L} = \exp^{-1}(\{e\})$. The crucial property of this map is that

$$(\tilde{F})\hat{\ }(\lambda) = C\hat{F}(\lambda + \beta), \qquad \lambda \in \hat{G},$$

where the Fourier transform of an invariant function f on G is given by

$$\hat{f}(\lambda) = (d_\lambda)^{-1} \int_G f(g) \overline{\chi_\lambda(g)}\, dg,$$

d_λ and χ_λ denoting, respectively, the dimension and character of λ.

Moreover, if $F \in L_i^1(\mathfrak{g}) \cap L_i^p(\mathfrak{g})$, then $\tilde{F} \in L_i^p(G)$ and $\|\tilde{F}\|_p \leqslant A\|F\|_p$, as long as

$$\sum_{\mathfrak{L}} |\Delta_0(X + l)|^{(2-p)/(p-1)} \leqslant B, \qquad X \in \mathfrak{h}. \tag{1}$$

3. **Proof of the Theorem**. We begin by establishing the desired result for exponents p satisfying (1) and groups G having what we call Propeerty A. (Although SU(n) and the groups it covers are the only simple compact groups having Property A, the property seems worth singling out.) The proof of the Theorem is completed by verifying that SU(n) does have Property A and that if $G = \mathrm{SU}(n)$, then (1) is satisfied for all p such that

$$1 \leqslant p < 2\left[1 - (n+2)^{-1}\right].$$

With notation as above and $Z(G)$ denoting the center of G, we can state

PROPERTY A. There exists $\varepsilon > 0$ such that for every $X \in \mathfrak{h}$ there is a $Z \in \mathfrak{h}$ such that $\exp Z \in Z(G)$ and $|\langle X + Z, \alpha_j \rangle| \leqslant 2\pi(1 - \varepsilon), j = 1, \ldots, r$.

Notice that if G has Property A, then $G = \bigcup_{k=1}^{K} N_k$, where the N_k are invariant, N_1 is a neighborhood of e consisting of points of the form $\exp X$ with $|\langle X, \alpha_j \rangle| \leqslant 2\pi(1 - \varepsilon)$, $j = 1, \ldots, r$, and for $2 \leqslant k \leqslant K$, there is a $z_k \in Z(G)$ with $z_k N_k \subset N_1$.

Also, if G has Property A so does any group covered by G. (Of course, Property A can be more simply recast in terms of the center-free group with

Lie algebra $\mathfrak{g}$, but SU(n), for example, is more often considered than its center-free local isomorph.)

PROPOSITION *Suppose that G has Property* A *and that p satisfies* (1). *Then the Theorem holds for G and p.*

PROOF. A standard limiting argument, which we merely outline, shows that m can be taken to be the Fourier transform of some $M \in L_i^1(\mathfrak{g})$. In fact, if $\{\varphi_\varepsilon\}$ is a suitable approximation to the identity in $L_i^1(\mathfrak{g})$, then $m_\varepsilon = \hat{\varphi}_\varepsilon \cdot (\varphi_\varepsilon * m)$ is of the form $\hat{M}_\varepsilon$, $M_\varepsilon \in L_i^1(\mathfrak{g})$, and it is easy to see that $\|T_{m_\varepsilon}\| \leqslant \|T_m\|$. Moreover, it follows from the continuity property hypothesized for m and the below that $T_{\tilde{m}} f = \lim_\varepsilon T_{(m_\varepsilon)^\sim} f$ pointwise for every f that is a finite linear combination of characters. Finally an application of Fatou's lemma yields the desired result for m from the same result for the m_ε.

Suppose now that $m = \hat{M}$, $M \in L_i^1(\mathfrak{g})$, and that $f \in L_i^p(G)$. It is enough to assume f to be bounded; we make the temporary assumption that in addition f is supported on $N = \exp \mathfrak{n}$, where $\mathfrak{n}$ is an invariant neighborhood of 0 in $\mathfrak{g}$ contained in a fundamental domain of exp and $|\langle X, \alpha_j\rangle| \leqslant 2\pi(1 - \varepsilon)$, $j = 1, \ldots, r$, $X \in \mathfrak{n}$. Notice that for $X \in \mathfrak{n}$, $|\Delta_0(X)| \geqslant C_\varepsilon$.

Now define F on $\mathfrak{g}$ by setting $F(X) = f(\exp X)\Delta_0(X)$, $X \in \mathfrak{n}$, $F(X) = 0$, otherwise; it follows that $\tilde{F} = f$. Moreover, letting $\mathfrak{q} = \mathfrak{n} \cap \mathfrak{h}$ we have (see [5])

$$\begin{aligned}\int_{\mathfrak{g}} |F(X)|^p\, dX &= \int_{\mathfrak{q}} |F(X)|^p |P(X)|^2\, dX \\ &= \int_{\mathfrak{q}} |f(\exp X)|^p |\Delta_0(X)|^{-2+p} |\Delta(X)|^2\, dX \\ &\leqslant B_\varepsilon^p \int_{\mathfrak{q}} |f(\exp X)|^p |\Delta(X)|^2\, dX = B_\varepsilon^p \int_G |f(g)|^p\, dg,\end{aligned}$$

where the inequality follows from the lower bound on $|\Delta_0|$.

Because of our assumptions on m and f, $T_m F = M * F$ is in $L_i^1(\mathfrak{g})$, and so for $\lambda \in \hat{G}$,

$$[(T_m F)^\sim]^\wedge(\lambda) = [T_m F]^\wedge(\lambda + \beta) = m(\lambda + \beta)\hat{F}(\lambda + \beta) = \tilde{m}(\lambda)\hat{f}(\lambda),$$

from which it follows that $(T_m F)^\sim = T_{\tilde{m}} f$. Therefore, taking into account that p satisfies (1), we have

$$\begin{aligned}\|T_{\tilde{m}} f\|_p &\leqslant A \|T_m F\|_p \leqslant A N(m; \mathfrak{M}_i^p(\mathfrak{g})) \|F\|_p \\ &\leqslant A B_\varepsilon N(m; \mathfrak{M}_i^p(\mathfrak{g})) \|f\|_p.\end{aligned}$$

The temporary assumption about the support of f is dropped by appealing to the consequence of Property A noted above. In particular, $G = \bigcup_{k=1}^K N_k$, where $N_1 = N$, the N_k are invariant and $z_k N_k \subset N$ for some $z_k \in Z(G)$ for $k \geqslant 2$. Notice that nothing is lost if the N_k are taken to be disjoint, which we do. Now set

$$f_k = f\chi_{N_k}, \quad g_k(x) = \lambda_{z_k} f_k(x) = f_k(z_k^{-1}x).$$

Clearly $f = \sum_{k=1}^K f_k$ and g_k is supported in $z_k N_k \subset N$; moreover; since $z_k \in Z(G)$, g_k is also invariant. The operators induced by multipliers commute with translation, so letting $T = T_{\tilde{m}}$ and applying what we have already proved to the g_k, we have

$$\|Tf_k\|_p = \left\|T(\lambda_{z_k})^{-1} g_k\right\|_p = \left\|(\lambda_{z_k})^{-1} T g_k\right\|_p = \|Tg_k\|_p$$
$$\leqslant B\|g_k\|_p = B\|f_k\|_p,$$

with $B = CN(m;\ \mathfrak{M}_i^p(\mathfrak{g}))$. Finally, because the f_k have disjoint supports, it follows that

$$\|Tf\|_p \leqslant B \sum_{k=1}^{K} \|f_k\|_p \leqslant BK^{(1-1/p)}\|f\|_p.$$

Turning to the proof that SU(n) has Property A, we realize $\mathfrak{h}$ as $\{X = (X_1, \ldots, X_n)\colon X_1 + \cdots + X_n = 0\}$, in which case the roots, considered now as members of $\mathfrak{h}^*$, are of the form

$$\alpha(X) = X_i - X_j.$$

Also, $\exp^{-1}(Z(\mathrm{SU}(n)))$ contains $W_j = 2\pi(n^{-1}, \ldots, n^{-1} - 1, \ldots, n^{-1}), j = 1, \ldots, n$, where the $n^{-1} - 1$ is in the jth place. Notice that for every α and X, $\alpha(X + W_j) = \alpha(X + U_j)$, where $U_j = 2\pi(0, \ldots, -1, \ldots, 0)$. (Of course, $X + U_j \not\in \mathfrak{h}$, but $\alpha(X + U_j)$ is still defined.) We are thus reduced to showing that given any $X \in \mathbf{R}^n$, we can, by translating each X_j by $\pm 2\pi$ the appropriate number of times, obtain the inequality $\max_{i,j}|X_i - X_j| \leqslant 2\pi(1 - \varepsilon)$.

We establish this inequality, with $\varepsilon = 2^{1-n}$, by induction. The case $n = 2$ is immediate. Assuming the inequality to hold for n, we can, given $X = (X_1, \ldots, X_{n+1})$, suppose after possible renumbering that $X_1 \leqslant \cdots \leqslant X_n$, $X_n - X_1 \leqslant 2\pi(1 - 2^{1-n})$. By translating X_{n+1}, we can also suppose that $0 \leqslant X_{n+1} - X_1 < 2\pi$. Now if $X_{n+1} - X_1 \leqslant 2\pi(1 - 2^{-n})$, we are done; otherwise,

$$\begin{aligned} X_n - (X_{n+1} - 2\pi) &= (X_n - X_1) - (X_{n+1} - X_1) + 2\pi \\ &< 2\pi(1 - 2^{1-n}) - 2\pi(1 - 2^{-n}) + 2\pi \\ &= 2\pi(1 - 2^{-n}), \end{aligned}$$

and the desired inequality again holds. (A slightly more complicated argument yields the best possible value $\varepsilon = 1/n$.)

We conclude the proof of the Theorem by showing that if $G = \mathrm{SU}(n)$, then (1) is satisfied for $p < 2(1 - (n+2)^{-1})$. Letting $D(l) = |\Pi\langle l, \alpha_j\rangle|$ for $l \in \mathfrak{L}$, where the product is taken over j such that $\langle l, \alpha_j\rangle \neq 0$, it is easy to see that for X in a fundamental domain in $\mathfrak{h}$, $|\Delta_0(X + l)| \leqslant AD(l)^{-1}$, and so we are reduced to establishing (1) with $|\Delta_0(X + l)|$ replaced by $D(l)^{-1}$.

We show that $\Sigma_{\mathfrak{L}} D(l)^{-\gamma} < \infty$ if $\gamma > 2/n$, which is enough since

$$(2 - p)/(p - 1) > 2/n$$

if $p < 2(1 - (n+2)^{-1})$. If we choose a basis for $\mathfrak{L}$ dual to the simple positive roots $\alpha_j(X) = X_{j+1} - X_j$ and restrict ourselves to the intersection of $\mathfrak{L}$ with the closure of the fundamental Weyl chamber (which we may), the summation over $\mathfrak{L}$ becomes a summation over $\mathfrak{L}^+$, the space of $(n-1)$-tuples of nonnegative integral multiples of 2π, and the positive roots are of the form

$$\alpha(l_1, \ldots, l_{n-1}) = l_j + l_{j+1} + \cdots + l_{j+k}, \qquad 1 \leq j \leq j + k \leq n - 1.$$

The proof that $\sum_{\mathcal{L}^+} D(l)^{-\gamma} < \infty$ if $\gamma > 2/n$ is by induction in n on the estimates

$$\sum_{(N)} D(l)^{-\gamma} \leq \begin{cases} C, & \gamma > 2/n, \\ C \log N, & \gamma = 2/n, \\ CN^{(n-1)(1-\gamma n/2)}, & \gamma < 2/n, \end{cases}$$

where $(N) = \{l \in \mathcal{L}^+ : 0 \leq l_j \leq N, j = 1, \ldots, n - 1\}$. These estimates are immediate when $n = 2$. Assuming the estimates for n, one obtains them for $n + 1$ by considering separately the cases $\gamma < 2/(n + 1)$, $\gamma = 2/(n + 1)$, $2/(n + 1) < \gamma < 2/n$, $\gamma = 2/n$, $\gamma > 2/n$, and making use of the fact that if $l_j > l_k$ for all $k \neq j$ then $D(l_1, \ldots, l_n) \geq l_j^n D(l_1, \ldots, \hat{l}_j, \ldots, l_n)$.

This completes the proof that (1) holds if $p < 2(1 - (n + 2)^{-1})$, and thus the proof of the Theorem.

REMARKS. 1. The condition that m is continuous at $\lambda + \beta$ can be replaced by the condition $m(\lambda + \beta) = \lim \varphi_\varepsilon * m(\lambda + \beta)$, which holds for suitable $\{\varphi_\varepsilon\}$ if $\lambda + \beta$ is a Lebesgue point of m.

2. The Theorem is trivially true with L_i^p replaced by L^2; it is shown in [5] that the Theorem is also true if L_i^p is replaced by L^1.

3. The critical exponent

$$\begin{aligned} p_G &= 2\left[1 - (n + 2)^{-1}\right] = 2(n^2 - 1)/(n^2 + n - 2) \\ &= 2 \dim G/(\dim G + \operatorname{rank} G) \end{aligned}$$

which arises here for $G = \mathrm{SU}(n)$ arises in [4] for arbitrary compact G; it is shown in [4] that norm convergence of Fourier series by certain summability methods fails for at least one $f \in L_i^p(G)$ if $p < p_G$ and succeeds for all $f \in L_i^p(G)$ for some p in the range $p_G < p \leq 2$.

4. It is easy to see that no simple Lie group with Lie algebra other than $\mathfrak{su}(n)$ has Property A; the point is that only in $\mathfrak{su}(n)$ is it true that every positive root is a linear combination of simple positive roots in which the only coefficients are 0 and 1.

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