PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

EXISTENCE OF A FIXED POINT FOR NONEXPANSIVE MAPPINGS WITH CLOSED VALUES

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ABSTRACT. Fixed point existence and fixed point stability results are presented for nonexpansive mappings of a Banach space B into the family of nonempty closed bounded convex subsets of B, where B is assumed separable, strictly convex, and reflexive with a weakly continuous duality mapping.

The study of the existence of a fixed point for nonexpansive set valued mappings was initiated in [7] for Hilbert space and extended in [6] to Banach spaces satisfying Opial's condition and in [5] to strictly convex reflexive Banach spaces with weakly continuous duality mapping. All of these results have assumed the mappings have compact values. In Theorem 1 the nonexpansive mapping is assumed to have closed bounded convex values and existence of a fixed point is shown for separable strictly convex reflexive Banach spaces with weakly continuous duality mapping. This class of spaces includes separable Hilbert spaces and the l_p spaces, $1 < p < \infty$.

The family of nonempty closed bounded convex subsets of a Banach space B is denoted by $K(B)$. Let D denote the Hausdorff metric defined on the closed bounded subsets of B, which is generated by the norm $\|\cdot\|$ of B. A mapping F of B into $K(B)$ is nonexpansive if $D(F(x), F(y)) \leqslant \|x - y\|$ for $x, y \in B$.

A mapping J of a Banach space B into its dual B^* is a duality mapping if $(x, J(x)) = \|x\| \, \|J(x)\|$ and $\|J(x)\| = \mu(\|x\|)$ for $x \in B$, where μ is a nonnegative nondecreasing function on R^1 with $\mu(0) = 0$. A duality mapping J is said to be weakly continuous if it is continuous from B with the weak topology into B^* with the weak*-topology. Weak convergence of a sequence $\{x_i\}$ to a point x is denoted by $x_i \rightharpoonup x$.

A mapping F of a Banach space B into itself is J-monotone provided for any pair $x, y \in B$ and $x_1 \in F(x)$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x - y)) \geqslant 0$, where J is a duality mapping on B.

For any mapping F of B into the nonempty subsets of B and any subset C of B, $F(C)$ denotes $\cup_{x \in C} F(x)$. A point $y \in B$ is a fixed point of F if $y \in F(y)$.

Received by the editors October 6, 1975 and, in revised form, February 24, 1976.
AMS (MOS) subject classifications (1970). Primary 47H10; Secondary 54C60.
Key words and phrases. Fixed point, set valued map, monotone map.

The following two lemmas extend similar results in [5] for compact valued mappings.

LEMMA 1. *Let B be a strictly convex reflexive Banach space with a weakly continuous duality mapping J, and F a continuous mapping of B with the norm topology into $K(B)$ with the Hausdorff metric D. If for a given pair x, $x_1 \in B$ and any $y \in B$ there is a $y_1 \in F(y)$ such that $(x_1 - y_1, J(x-y)) \geqslant 0$, then $x_1 \in F(x)$.*

PROOF. Let x, x_1 be elements of B such that for any $y \in B$ there is $y_1 \in F(y)$ satisfying $(y_1 - x_1, J(y-x)) \geqslant 0$. Suppose $x_1 \notin F(x)$. Since $F(x)$ is weakly compact and convex there is a continuous linear functional w strictly separating x_1 and $F(x)$; i.e., $(x_1, w) < (z, w)$ for $z \in F(x)$. Ko [5] has shown that if B is reflexive then a weakly continuous duality mapping maps B onto B^* and therefore $w = J(u)$ for some $u \in B$. Hence,

$$(x_1, J(u)) < (z, J(u)) \tag{1}$$

for $z \in F(x)$.

Setting $u_n = x - u/n$, $n = 1, 2, \ldots$, there is by assumption for each u_n a $z_n \in F(u_n)$ such that

$$(x_1 - z_n, J(x-u_n)) = (x_1 - z_n, J(u/n)) \geqslant 0. \tag{2}$$

By a result of Browder [3] the strict convexity of the norm of B implies that $J(u/n) = J(u)/n$. Inequality (2) can then be written as

$$(x_1 - z_n, J(u)) \geqslant 0 \tag{3}$$

for each n.

By the continuity of F, $D(F(u_n), F(x))$ tends to 0 and therefore we may assume that $\{z_n\}$ converges weakly to a point z_0 and that there is a sequence $\{y_n\}$, $y_n \in F(x)$ for which $\lim_{n\to\infty} \|z_n - y_n\| = 0$, where $y_n \rightharpoonup y_0 \in F(x)$. We assert that $z_0 \in F(x)$ so that by inequality (3), $(x_1, J(u)) \geqslant (z_0, J(u))$, contradicting (1). Indeed, if $z_0 \notin F(x)$ there is a continuous linear functional v such that $(z_0, v) < (y_0, v)$ and hence

$$0 > (z_0 - y_0, v) = (z_0 - z_n, v) + (z_n - y_n, v) + (y_n - y_0, v).$$

The right side of the latter equality tends to 0, which is not possible.

LEMMA 2. *Let B be a separable strictly convex reflexive Banach space with weakly continuous duality mapping J, C a weakly compact subset of B, and F a continuous J-monotone mapping of B into $K(B)$ with the Hausdorff metric D. Then $F(C)$ is closed.*

PROOF. Let v_0 lie in the closure of $F(C)$. Then there is a sequence $\{v_i\}$ such that $\lim_{i\to\infty} v_i = v_0$, where $v_i \in F(u_i)$, $u_i \in C$ and, by weak compactness of C, it is assumed that $u_i \rightharpoonup u_0 \in C$. The assumption that $v_0 \notin F(C)$ will be shown to lead to a contradiction.

For some $x \in B$ it must be the case that there is a $\delta > 0$ such that

$$(4) \qquad (z - v_0, J(x - u_0)) < -\delta$$

for each $z \in F(x)$; for otherwise Lemma 1 would imply that $v_0 \in F(u_0)$. For each nonnegative integer j let $B_j = F(x) - v_j$. Since $v_j \to v_0$, the sequence $\{B_j\}$ converges to B_0 in the Hausdorff metric D.

Choose a closed ball S in B which contains the sets $\{B_j\}, j = 0, 1, \ldots$. By the reflexivity of B the ball S is weakly compact and by [4] the weak topology on S is metrizable. Metrizing the weak topology on S by

$$d(a,b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n(a-b)|}{1 + |x_n(a-b)|}$$

where $\{x_n\}$ is a countable dense subset of the unit ball of B^* and $a, b \in S$, it is easily seen that $\{B_j\}$ converges to B_0 in the Hausdorff metric H generated by d. Indeed,

$$d(a,b) \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|a-b\|}{1 + \|a-b\|} \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) \|a-b\|$$

implying that

$$H(A,B) \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) D(A,B),$$

where A, B are weakly closed subsets of S.

Define the functionals $\{L_i\}$ on B by $L_i(y) = \sup_{j \geqslant i}(y, J(x - u_j)), y \in B$. Since the functionals $\{(\cdot, J(x - u_j))\}$ are linear and, hence, convex, each L_i is a convex functional [2].

Let the function L denote an arbitrary member of $\{L_i\}$. We claim that L is continuous on S with the weak relative topology. Since L is bounded on any norm bounded subset of B, assume that $L(y) \leqslant K$ for any $y \in S$. Let $N(0, \delta)$ be an origin centered open ball of radius δ in the weak metric topology on S. For $\alpha \in [0, 1)$ it is easily seen that $\alpha N(0, \delta) \subseteq N(0, \delta)$. If $y \in \alpha N(0, \delta)$ then there is a $z \in N(0, \delta)$ for which $y = \alpha z$ and therefore $(y/\alpha) \in N(0, \delta)$. By the convexity of L,

$$L((1 - \alpha)0 + \alpha(y/\alpha)) \leqslant (1 - \alpha)L(0) + \alpha L(y/\alpha).$$

Since $L(0) = 0$, $L(y) \leqslant \alpha L(y/\alpha) \leqslant \alpha K$. Hence, in the limit as y approaches 0 in the weak topology $L(y) \leqslant L(0)$, which proves that L is upper semicontinuous at 0 in the weak topology on S. Continuity of L at 0 follows from the lower semicontinuity of the supremum of continuous functions and the definition of L. For any other point $y \in S$ appropriate translations reduce the problem to the case just considered where $y = 0$ and $L(y) = 0$.

Since F is J-monotone, for each u_j there is a $z_j \in F(x)$ such that $(z_j - v_j, J(x - u_j)) \geqslant 0$. Thus, for each $j \geqslant i$, $\sup_{y \in B_j} L_i(y) \geqslant 0$. The maximum theorem in [1] implies that

(5) $$\lim_{j\to\infty} \sup_{y\in B_j} L_i(y) = \sup_{y\in B_0} L_i(y) \geqslant 0 \quad \text{for } i = 1, 2, \ldots.$$

For each positive integer i let $A_i = \{y \in B_0 \colon L_i(y) \geqslant -1/i\}$. If $y \in A_{i+1}$ then, by the definition of the L_i, $L_i(y) \geqslant L_{i+1}(y) \geqslant -1/(i+1) > -1/i$, and therefore, $y \in A_i$. Thus, $A_{i+1} \subseteq A_i$ and, since the A_i are weakly closed subsets of the weakly compact set B_0, there is a point $y_0 \in \bigcap_{i=1}^{\infty} A_i$. By the definition of B_0, $y_0 = z_0 - v_0$ where $z_0 \in F(x)$, and by (5) for some subsequence $\{u_k\}$ of $\{u_i\}$, $(z_0 - v_0, J(x - u_k)) \geqslant -1/k$. Taking the limit in the latter inequality we have $(z_0 - v_0, J(x - u_0)) \geqslant 0$, which contradicts (4).

THEOREM 1. *Let B be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping J, and C a closed bounded convex subset of B. If G is a nonexpansive mapping of B into $K(B)$ with the Hausdorff metric D, which maps C into itself, then G has a fixed point in C.*

PROOF. Assume without loss of generality that $0 \in C$ [5]. The proof consists of showing that 0 is in the closure of $(I - G)(C)$ and that $I - G$ is J-monotone. The theorem then follows by Lemma 2.

Let $\{k_i\} \subseteq [0, 1)$ be a sequence which converges to 1, and consider the sequence of mappings $\{k_i G\}$ of C into $K(C)$. By a result of Nadler [8] each $k_i G$ has a fixed point $x_{k_i} \in C$. Since $x_{k_i} \in k_i G(x_{k_i})$ we have $x_{k_i} = k_i y_{k_i}$, where $y_{k_i} \in G(x_{k_i})$. Therefore,

$$\inf_{y\in G(x_{k_i})} \|y - x_{k_i}\| \leqslant \|x_{k_i} - y_{k_i}\| \leqslant (1 - k_i)\|y_{k_i}\|$$

and the last term tends to 0. This shows that 0 is in the closure of $(I - G)(C)$.

G being nonexpansive with closed convex values, given any $y \in B$ and $y_1 \in G(y)$, there is a closest point $x_1 \in G(x)$ to y_1 such that $\|x_1 - y_1\| \leqslant \|x - y\|$. It follows that

$$((x - x_1) - (y - y_1), J(x - y)) \geqslant (\|x - y\| - \|x_1 - y_1\|)\|J(x - y)\| \geqslant 0,$$

and hence $I - G$ is J-monotone. Applying Lemma 2 we have $0 \in (I - G) \cdot (C)$; i.e., there is an $x \in C$ such that $x \in G(x)$.

THEOREM 2. *Let B be a separable strictly convex reflexive Banach space with a weakly continuous duality mapping J, and C a closed bounded convex subset of B. Assume that $\{G_i\}$ is a sequence of nonexpansive mappings of B into $K(B)$ with the Hausdorff metric, which converges pointwise to a nonexpansive mapping G_0 and maps C into itself. If $x_i \in C$ is a fixed point of G_i, $i = 1, 2, \ldots$, and $x_i \rightharpoonup x_0$ then x_0 is a fixed point of G_0.*

PROOF. As in the proof of Theorem 1 the mappings $I - G_i$, $i = 0, 1, \ldots$, are J-monotone. Since x_i is a fixed point of G_i, $0 \in (I - G_i)(x_i)$ for $i = 1, 2, \ldots$, and by J-monotonicity for each $v \in B$ there is a $v_i \in (I - G_i)(v)$ for which

(6) $$(v_i - 0, J(v - x_i)) \geqslant 0, \qquad i = 1, 2, \ldots.$$

Define the sequence of functionals $\{L_i\}$ on B by

$$L_i(y) = \sup_{j \geqslant i}(y, J(v - x_j)).$$

As in the proof of Lemma 2 the $\{L_i\}$ are continuous on S with the weak topology. Defining the sequence of weakly compact convex subsets $\{B_j\}$ of B by $B_j = (I - G_j)(v)$, the pointwise convergence of the $\{G_j\}$ implies that $\{B_j\}$ converges to $B_0 = (I - G_0)(v)$ in the Hausdorff metric D, and therefore, as shown in the proof of Lemma 2, the $\{B_j\}$ converge to B_0 in the Hausdorff metric H generated by the weak metric topology on any ball containing the $\{B_j\}$. Inequality (6) implies that for each i, $\sup_{y \in B_j} L_i(y) \geqslant 0$ for $j \geqslant i$.

Thus, the sequences $\{L_i\}$ and $\{B_j\}$ satisfy the same conditions as in the proof of Lemma 2, and therefore there is a point $y_0 \in B_0$ such that $(y_0 - 0, J(v - x_0)) \geqslant 0$. The point $v \in B$ was arbitrary and so by Lemma 1, $0 \in (I - G_0)(x_0)$ and x_0 is a fixed point of G_0.

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