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A NOTE ON UNCONDITIONALLY CONVERGING SERIES IN L_p

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ABSTRACT. THEOREM. *A series* $\sum f_i$ *in* $L_p[0,1]$ $(1 \leqslant p \leqslant 2)$ *is unconditionally convergent if and only if for each i and for all* $t \in [0,1]$, $f_i(t) = \alpha_i g(t) w_i(t)$ *where* $(\alpha_i) \in l_2$, $g \in L_2[0,1]$ *and* (w_i) *is an orthonormal sequence in* $L_2[0,2]$. *This characterization allows the generalization (to u.c. series in* $L_p[0,1]$*) of several classical theorems concerning almost everywhere convergence of orthogonal series in* L_2.

Recently, Maurey and Nahoum [7] and Bennett [1] have generalized certain classical results of Menšov and Kolmogoroff and an interesting result of Garsia on almost everywhere convergence of orthogonal series in $L_2[0,1]$ to similar results for almost everywhere convergence of unconditionally convergent series in $L_p[0,1]$. In an effort to find a principle underlying such generalizations, we were led to the theorem stated in the abstract which reduces certain questions of a.e. convergence for u.c. series in L_p to the same questions for orthogonal series.

Recall that a series $\sum f_i$ in X is *unconditionally convergent* if the map $T(\alpha_i) = \sum \alpha_i f_i$ is continuous as an operator from l_∞ to X. Further, since the natural injection of L_p into L_1 is continuous when $p \geqslant 1$, an unconditionally convergent series in L_p with $p \geqslant 1$ may as well be considered to be an unconditionally convergent series in L_1. We now are in a position to prove the theorem stated in the abstract.

Let $\sum f_i$ be u.c. in $L_1[0,1]$ and let $T\colon l_\infty \to L_1$ be the map defined above. According to a result of Grothendieck [3] (see also [5]) any such T must factor through a Hilbert space. By successive applications of the Grothendieck-Pietsch factorization theorem (see [3], [10], [12]) T admits a factorization of the form $T = T_g AB$. Here $(T_g h)(t) = g(t)h(t)$ with g in $L_2[0,1]$. That is, setting $h_i = f_i/g$, $\sum h_i$ is u.c. in L_2. Also there exists $(\mu_i) \in l_1$, $\mu_i \geqslant 0$, $\sum \mu_i = 1$ so that $B\colon l_\infty \to l_2(\mu_i)$ is the natural basis-basis map, and $A\colon l_2(\mu_i) \to L_2[0,1]$ takes the natural basis in $l_2(\mu_i)$ to h_i. We have a constant $K > 1$ so that $\|\sum c_i h_i\|_2 \leqslant K(\sum \mu_i hc_i h^2)^{1/2}$. Therefore, for any orthonormal sequence (φ_n) in $L_2[0,1]$, the map defined by $C\varphi_n = K^{-1}\mu_n^{-1/2} h_n$ defines a contraction on $L_2[0,1]$. By Nagy's dilation theorem (e.g. [9]), there is a Hilbert space H and

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a unitary operation U on $L_2[0,1] \oplus H$ so that $Ch = RUh$ for all h in $L_2[0,1]$. Letting $H = L_2[1,2]$, U is unitary on $L_2[0,2]$ and R is the natural restriction operator from $L_2[0,2]$ to $L_2[0,1]$. Setting $w_n = U\varphi_n$ and $\alpha_n = K\sqrt{\mu_n}$, we have the desired result.

Since the maps R and T_g preserve almost everywhere convergence, we get the following generalizations [1], [7] of the theorems of Menšov [8], [11], Kolmogoroff (e.g. [4, p. 23]) and Garsia [2] (respectively).

COROLLARY. (a) *If* $\sum f_i$ *is u.c. in* $L_p[0,1]$, *then* $\sum f_i/\log(i+1)$ *converges a.e.*

(b) *If* $\sum f_i$ *is u.c. in* $L_p[0,1]$, *then for almost all choices of* $(\varepsilon_i) = (\pm 1)$, *the series* $\sum \varepsilon_i f_i$ *converges a.e.*

(c) *If* $\sum f_i$ *is u.c. in* $L_p[0,1]$, *there is a permutation* π *of the integers so that* $\sum f_{\pi(i)}$ *converges a.e.*

REMARK. For other purposes, a slight strengthening of the theorem is possible. For $\sum f_i$ u.c. in L_p with $1 < p < 2$ the function g may be chosen in $L_r[0,1]$ where $1/2 + 1/r = 1/p$ according to the extension of Grothendieck's theorem due to Rosenthal and Maurey (e.g. [12], [6]).

ADDED IN PROOF. (*) If $\sum f_i$ is u.c. in $L_1[0,1]$, then for every $\varepsilon > 0$ there is a set E_ε, $m(E_\varepsilon) > 1 - \varepsilon$ such that $\sum f_i|_{E_\varepsilon}$ converges unconditionally in $L_p(E_\varepsilon)$ for every $p \leqslant 2$.

PROOF. Let g be as in the Theorem stated in the abstract. For any $\varepsilon > 0$, pick M so that $\{t \mid |g(t)| \leqslant M\} = E_\varepsilon$ has $m(E_\varepsilon) > 1 - \varepsilon$. Then $g^M(t) \equiv g(t)\chi_{E_\varepsilon}$ is in L_∞, so that $\sum g^M(t)\alpha_i\varphi_i(t) = \sum f_i(t)\chi_{E_\varepsilon}$ converges unconditionally in L_2.

This answers a question of E. M. Nikishin and strengthens a result of Kašin [13] who proved (*) for $p < 2$.

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