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INJECTIVE COGENERATOR RINGS AND A THEOREM OF TACHIKAWA[1]

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For Seth Camillo, and the happy parents.

ABSTRACT. Tachikawa showed that a left perfect ring R is an injective cogenerator in the category of all right R-modules iff there holds: **(right FPF)** every finitely generated faithful right module generates mod-R. In this paper, we simplify Tachikawa's long and difficult proof by first obtaining some new structure theorems for a *general semiperfect right FPF ring* R; the most important are: *R is a direct sum of uniform right ideals, and every nonzero right ideal of the basic ring R_0 of R contains a nonzero ideal of R_0. Furthermore, if the Jacobson radical* $\operatorname{rad} R$ *is nil, then R is right self-injective.* Tachikawa's theorem is an immediate consequence. We also generalize a theorem of Osofsky on perfect PF rings to FPF rings.

1. **Introduction.** A ring R is said to be **right (F)PF** provided that every (finitely generated) faithful right module M generates the category mod-R of all right R-modules. Theorems of Azumaya [66], Osofsky [66], and Utumi [67] characterize a right PF ring by the equivalent conditions.

(PF_1) *R is right self-injective and semiperfect with essential right socle.* (*The socle is the largest semisimple submodule.*)

(PF_2) *R is right self-injective with finite essential right socle.*

(PF_3) *R is a finite direct sum, $R = \sum_{i=1}^{n} \oplus e_i R$, where $e_i^2 = e_i \in R$ and $e_i R$ is a projective injective right ideal with simple socle, $i = 1, \ldots, n$.*

(PF_4) *R is an injective cogenerator in* mod-R.

(PF_5) *R is right self-injective and every simple right module embeds in R.*

(Compare Kato [68] and Onodera [68].)

These rings generalize the Quasi-Frobenius (QF) rings of Nakayama (the Artinian PF rings), and the twosided PF rings of Morita [58]. The latter rings possess a duality between the reflexive right R-modules and the reflexive left R-modules induced by $\operatorname{Hom}_R(\ ,R)$. A theorem of Tachikawa [69] establishes that any left perfect (in the sense of Bass [60]) right FPF ring R is actually right PF; consequently right Artinian right or left FPF rings are QF. In this paper we generalize Tachikawa's theorem (in (2) of Theorem 1) and in doing so obtain a simpler (and self-contained) proof. (Most undefined terms are explained in §2. Also, see Faith [76a], especially Chapter 24.)

1. THEOREM. (1) *If R is a semiperfect right FPF ring, then R is a direct sum of uniform right prindecs* (= *principal indecomposable right ideals*), *and every*

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nonzero right ideal of the basic ring R_0 of R contains a nonzero ideal of R_0. (2) *Moreover, if* rad R *is a nil ideal, then R is right self-injective.*

2. THEOREM (CONVERSE OF THEOREM 1). *Any right self-injective semiperfect ring R is right FPF if every nonzero right ideal of the basic ring R_0 contains an ideal of R_0.*

Theorem 1 implies Tachikawa's theorem (via PF_1) since any left perfect ring has nil radical and essential right socle (Bass [60]). Incidentally this proves

3. COROLLARY. *Any right (or left) perfect right FPF ring is right self-injective.*

We also prove

4. THEOREM. *A left FPF right PF ring with nil radical is left PF.*

5. COROLLARY. *A left perfect left and right FPF right is QF.*

The corollary generalizes the theorem of Osofsky [66] for one-sided perfect two-sided PF rings. The corresponding question for one- or even two-sided perfect one-sided PF rings is open.

2. **Background.** Before going to the proofs of the stated theorem, we supply the relevant background material for these.

2.1 DEFINITION AND PROPOSITION. *Let* mod-R *denote the category of right* R-modules *for a ring R. An object M of* mod-R *is a generator iff the equivalent conditions hold*:

G1. *The set-valued functor* $\mathrm{Hom}_R(M, \ )$ *is faithful.*

G2. *Given an object X of* mod-R, *there is an index set I and an exact sequence* $M^{(I)} \to X \to 0$, *where $M^{(I)}$ is the coproduct of I copies of M.*

G3. *There is a finite integer* $n > 0$, *an object Y of* mod-R, *and an isomorphism* $M^n \approx R \oplus Y$.

G4. *The trace ideal* $\mathrm{trace}_R\, M = \sum_{f \in \mathrm{Hom}_R(M,R)} f(M)$ *equals R.*

2.2 DEFINITION AND PROPOSITION (MORITA). *Let* R-mod *denote the left-right symmetry of* mod-R. *Two rings A and B are* **similar** *provided that the equivalent conditions hold*:

S1. mod-$A \approx$ mod-B.

S2. *There exists a finitely generated projective generator P of* mod-A *such that* $B \approx \mathrm{End}\, P_A$.

S3. A-mod $\approx$ *B-mod.*

In the case S2, $\mathrm{Hom}_A(P, \ )$ *induces an equivalence* mod-$A \approx$ mod-B *and the left adjoint $\otimes_B P$ is the inverse equivalence.* (The equivalence of S1–S3 is Morita's theorem [58]. Cf. Bass [68] or Faith [73, Theorem 4.29].)

A **Morita invariant property** of A is a property defined for the category of rings which A and any ring similar to A possesses.

Semiperfect rings. Let $R = \oplus_{i=1}^n e_i R$ be a direct sum decomposition of R into principal indecomposable right ideals $e_1 R, \ldots, e_n R$. By definition, then, e_i is an idempotent $\neq 0$, $e_i R e_i$ is a local ring, and $e_i R$ is an indecomposable right ideal, which we call a right prindec, for short, $i = 1, \ldots, n$. By a theorem of Bass [60], a ring R has such a decomposition if (and only if) R is **semiperfect** in the sense that $R/\mathrm{rad}\, R$ is semisimple, or, as we say, R is **semilocal**, and idempotents of R lift modulo rad R.

Basic modules and rings. Now assume the notation above. Renumber idempotents if necessary so that $e_1 R/e_1 J, \ldots, e_m R/e_m J$ constitute the isomorphism classes of simple right modules. Thus, every simple module $\approx$ some $e_i R/e_i J$, with $i \leqslant m$, and $e_i R/e_i J \approx e_k R/e_k J$ iff $i = k$, for all i and $k \leqslant m$. The right ideal $B = e_1 R + \cdots + e_m R$ is called the **basic right** module of R, $e_0 = e_1 + \cdots + e_m$ is then called the **basic idempotent**, and $e_0 R e_0 \approx \operatorname{End} B_R$ is the **basic ring** of R. The basic module is unique up to isomorphism, and if f_0 is any other basic idempotent, there is a unit x of R such that $f_0 = x e_0 x^{-1}$. Furthermore, B is the unique (up to isomorphism) minimal (finitely generated projective) generator of mod-R, and, in fact, if G is any other generator, there is a module X of mod-R, and an isomorphism $G \approx B \oplus X$. (This follows from the Krull-Schmidt theorem. See, e.g., Bass [68, p. 19, (3.5)].) Thus, if $M = M_1 \oplus \cdots \oplus M_n = A \oplus X$, for modules M_i, A, and X, $i = 1, \ldots, n$, where $\operatorname{End} A_R$ is a local ring, then, for some j, A is a direct summand of M_j. Thus, if each M_i is indecomposable, then $M_j = A$.

By the Morita theorem, every semiperfect ring R is similar to its basic ring. The ring R is said to be **self-basic** iff $R = B$. (This condition is right-left symmetric, inasmuch as R is self-basic iff $R/\operatorname{rad} R$ is a finite product of fields.) The basic ring of a ring is self-basic. Categorical properties such as R is right self-injective (= finitely generated projective right modules are injective) are Morita invariant properties, and therefore hold for R iff they hold for the basic ring of R.

The term **uniform** is used in Goldie's sense, namely, a right module (or right ideal) U is uniform iff $I \cap K = 0$ for two submodules I and K imply that $I = 0$ or $K = 0$. Equivalently, the injective hull $E = \hat{U}$ of U is indecomposable. (In this case, $B = \operatorname{End} E_R$ is a local ring by a theorem of Utumi [56].)

3. **Proofs of Theorems.** We begin with:

6. PROOF OF THEOREM 1. Let B be the basic right module, and let $B = e_1 R \oplus \cdots \oplus e_n R$ be its decomposition into a direct sum of mutually nonisomorphic right prindecs (see §2). Suppose, for example, that $I \cap K = 0$ for two submodules I and K of $e_1 R$. Then, $M = e_1 R/I \oplus e_1 R/K \oplus (1 - e_1)R$ is a faithful right module inasmuch as its annihilator ideal Q annihilates $(1 - e_1)R$ and $e_1 R$, that is, Q annihilates B, which is faithful. Since M therefore generates mod-R, we have $X \in$ mod-R such that $M \approx R \oplus X = e_1 R \oplus (1 - e_1)R \oplus X$. Inasmuch as $e_1 R/A$ and $e_2 R/B$ are indecomposable, and $e_i R$, $i = 1, \ldots, n$, all have local endomorphism rings, then by the Krull-Schmidt theorem cited above, necessarily $e_1 R \approx e_1 R/I$ or $e_1 R/K$. In the first case I splits via projectivity of $e_1 R$, so that indecomposability of $e_1 R$ implies that $I = e_1 R$, in which case $K = e_1 R \cap K = I \cap K = 0$, or $I = 0$. Thus $I = 0$ or $K = 0$ (also in case $e_1 R \approx e_1 R/K$), proving uniformity of $e_1 R$, and of $e_i R$, $i = 2, \ldots, n$, by symmetry. Since every right prindec eR is isomorphic to one of the $e_i R$, $i = 1, \ldots, n$, then every right prindec of R is uniform, as required.

To complete the proof of (1), we may assume R is self-basic. The top of any right R-module M is defined to be M/MJ, where $J = \operatorname{rad} R$, which is a semiperfect ring in the largest semisimple factor module. If I is a right ideal containing no ideals $\neq 0$, then the right module R/I is faithful, hence generates mod-R. Since R is self-basic, then $R/I \approx R \oplus X$ for some $X \in$ mod-R. Hence $|\text{top } R/I| = |\text{top } R| + |\text{top } X|$. But top $R/I = R/(I + J)$, that is, $|\text{top } R/I| \leqslant |\text{top } R| = |R/J|$, and top $X = X/XJ$. It follows that $I \subseteq J$, and $X = XJ$. Since X is finitely generalized, this implies $X = 0$, that is, $R/I \approx R$, so I splits. Since $I \subseteq J$, this implies $I = 0$.

(2) In order to prove that R is right self-injective, it suffices to prove that R_0 is. Moreover, R_0 is semiperfect with nil radical $e_0 J e_0$, where e_0 is the basic idempotent. Hence assume that R is self-basic.

Let u be an arbitrary element of the injective hull $e_1 R$ of $\widehat{e_1 R}$, and let $U = uR + e_1 R$. Then, $M = U + (1 - e_1)R$ is a faithful and finitely generated module, hence generates mod-R, so that $M \approx R \oplus X$ for some $X \in$ mod-R, that is,

$$M = (uR + e_1 R) \oplus e_2 R \oplus \cdots \oplus e_n R \approx e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R \oplus X.$$

Now, inasmuch as $e_i R e_i \approx \text{End } e_i R_R$ is a local ring, $i = 1, \ldots, n$, and $uR + e_1 R$ is indecomposable, the Krull-Schmidt theorem implies that $e_1 R$ is isomorphic to a direct summand of one of the direct summands of M. But, since R is self-basic, and $e_i R$ is indecomposable, then $e_1 R$ is not a direct summand of $e_i R$ for $i \geqslant 2$, hence $e_1 R$ is isomorphic to a direct summand at U. But U is uniform (contained in the injective hull of the uniform module $e_1 R$), so therefore $U = uR + e_1 R \approx e_1 R$. Hence $B = \text{End } U_R$ is a local ring $\approx e_1 R e_1$, and $Q = \text{rad } B$ is a nil ideal $\approx e_1 J e_1$. Therefore, since the endomorphism $f\colon U \to U$ induced by the isomorphism $U \to e_1 R$ has zero kernel, then f cannot lie in Q, that is, f cannot be nilpotent. Thus, f lies outside of Q, hence f is a unit, so that $U = f(U) = e_1 R$. Since this is true for all $u \in \widehat{e_1 R}$, then certainly $\widehat{e_1 R} = e_1 R$, so $e_1 R$ is injective. Similarly, $e_i R$ is injective, $i \geqslant 2$, and so then is R. □

7. Comment. It can be shown (Faith [7]) that any right FPF ring is **right bounded** in the sense that every essential right ideal contains a nonzero ideal. (Expressed otherwise: a cyclic module R/I is faithful only if I is inessential.) This generalizes part of Theorem 1 to nonsemiperfect rings.

8. Proof of Theorem 2. Let $M = x_1 R + \cdots + x_n R$ be a finitely generated faithful module. As in the proof of Theorem 1, we may suppose that R is self-basic, and therefore, the right ideal $K = \bigcap_{i=1}^{n} \text{ann } x_i$ is either 0, or else contains an ideal $B \neq 0$. But then

$$MB = \sum_{i=1}^{n} x_i RB = \sum_{i=1}^{n} x_i B = 0$$

contrary to the faithful hypothesis on M. Hence $\bigcap_{i=1}^{n} \text{ann } x_i = 0$, and therefore $R \hookrightarrow M^n$ canonically. Since R is right self-injective, then R splits in M^n, hence $M^n \approx R \oplus X$ for some $X \in$ mod-R, so therefore, M generates mod-R. This proves right FPF. □

We require the next lemma for the proof of Theorem 4. For a subset S of R, we let $S^{\perp} = \{r \in R | Sr = 0\}$, and ${}^{\perp}S$ its left-right symmetry.

9. Lemma. *Let R be a self-basic semiperfect ring in which ideals faithful on the right generate* mod-R. *Moreover, assume that an element c of R is right regular (in the sense that $c^{\perp} = 0$) only if $cR = R$. Then every simple left R-module V embeds in R.*

PROOF. Let V be a simple left R-module, and $P = \operatorname{ann}_R V$. Since R is semiperfect, then R/P is simple Artinian, and, since R is self-basic, R/P is a field. Thus, $V \approx R/P \hookrightarrow R$ iff there is an $x \in R$ with ${}^{\perp}x = P$. Since P is a maximal left ideal, this happens iff $P^{\perp} \neq 0$. Thus $V \not\hookrightarrow R \Rightarrow P^{\perp} = 0$, hence P generates mod-R by the hypothesis on ideals. Since R is self-basic, then $P \approx R \oplus X$ for some $X \in$ mod-R, and hence there exists $c \in P$ with $c^{\perp} = 0$. By the hypothesis on right regular elements, then $P = R$, a contradiction. This proves what we wanted. □

10. REMARKS. 1. Every right faithful ideal generates mod-R if R is right PF.

2. The hypothesis on c holds if either R is right self-injective, or left perfect.

1 is obvious from the definition, and in 2, the map $cR \to R$ sending $c \mapsto 1$ has an extension to an element $f \in \operatorname{End} R_R$, when R_R is injective, and then $y = f(1)$ satisfies $yc = 1$. Then R semiperfect (in fact the nonexistence of infinite sets of orthogonal idempotents suffices for this) yields $cy = 1$. Thus $cR = R$.

When R is left perfect, then there exist an integer n and an element $y \in R$ such that $c^{n+1}y = c^n$. (This by the d.c.c. on principal right ideals, e.g. on $\{c^n R\}_{n=1}^{\infty}$.) Then $c^{\perp} = 0 \Rightarrow cy = 1$ as before. □

11. COROLLARY (KATO [67]). *If R is right PF then every simple left module embeds in R.*

PROOF. Right $PF \Rightarrow$ semiperfect. We may assume that R is self-basic, and apply 9, using 10. □

12. COROLLARY (KATO [67]). *A right PF ring is left PF iff left self-injective.*

PROOF. Immediate from 11 and the left-right symmetry of (PF_5).

13. PROOF OF THEOREM 4. Immediate from 12 and Theorem 1.

14. PROOF OF COROLLARY 5. Apply Tachikawa's theorem to get one-sided PF, apply Theorem 4 to get two-sided PF, and then Osofsky's theorem yields QF (Osofsky [66]). □

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