PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

CONTINUOUS MAPPINGS FROM CANTOR SPACES ONTO INVERSE LIMIT SPECTRA

ALAN H. SCHOENFELD

ABSTRACT. Let $\mathbb{S} = \{X_\alpha; f_{\alpha\beta}: X_\beta \to X_\alpha\}$ be an inverse limit spectrum of compact Hausdorff spaces. We obtain necessary and sufficient conditions that there be a closed subspace W of a Cantor space and a family $\{f_\alpha: W \to X_\alpha\}$ of continuous surjections such that for each pair $\alpha < \beta$, $f_{\alpha\beta} \circ f_\beta = f_\alpha$. This result is applied to a special class of inverse spectra.

I. **Introduction.** It is well known that any compact Hausdorff space is the continuous image of a closed subspace of a Cantor space, any compact metric space the continuous image of 2^ω, the Cantor set. This paper was motivated by the following problem:

Let $\{X_i | i = 0, 1, 2\}$ be compact metric spaces and let $\{f_{0i}: X_i \to X_0 | i = 1, 2\}$ be given continuous surjections. Do there exist continuous surjections $\{f_i: 2^\omega \to X_i | i = 0, 1, 2\}$ such that $f_0 = f_{01} \circ f_1 = f_{02} \circ f_2$?

The answer (affirmative) follows easily from Theorem 3. Since the collections $\{X_i; f_{ij}\}$ form a (trivial) inverse spectrum, that result suggested that one might examine families of maps onto inverse spectra in general.

AGREEMENT. In this paper we assume each space X_α is nonempty, and that the bonding map $f_{\alpha\alpha}: X_\alpha \to X_\alpha$ is the identity on X_α for each α. The αth projection from $\prod_\alpha X_\alpha$ to X_α will be denoted p_α. We will denote $\varprojlim X_\alpha$ by X_∞.

II. **General maps onto inverse limit spectra.**

DEFINITION 1. Let $\mathbb{S} = \{X_\alpha; f_{\alpha\beta}: X_\beta \to X_\alpha\}$ be an inverse spectrum. Let S be a set and $\{f_\alpha: S \to X_\alpha\}$ a collection of set maps such that

(i) Each f_α is a surjection.

(ii) If $\alpha < \beta$, $f_\alpha = f_{\alpha\beta} \circ f_\beta$.

Then we say *the* $\{f_\alpha\}$ *map* S *onto* $\mathbb{S}$.

THEOREM 1. *Let* $\mathbb{S} = \{X_\alpha; f_{\alpha\beta}: X_\beta \to X_\alpha\}$ *be an inverse spectrum of compact Hausdorff spaces. Then* (A) *and* (B) *below are equivalent.*

(A) *There is a set* S *and a family of set maps* $\{F_\alpha: S \to X_\alpha\}$ *that map* S *onto* $\mathbb{S}$.

(B) *There is a closed subspace* W *of a Cantor space and a family of continuous surjections* $\{f_\alpha: W \to X_\alpha\}$ *that map* W *onto* $\mathbb{S}$.

PROOF. Clearly (B) implies (A). Assume (A), and consider the set

$$Y = \{\langle F_\alpha(s)\rangle | s \in S\} \subset \prod_\alpha X_\alpha.$$

Received by the editors December 26, 1974 and, in revised form, May 28, 1975.
AMS (MOS) subject classifications (1970). Primary 54C05, 54B25; Secondary 54C60.
Key words and phrases. Inverse limit spectra, Cantor spaces, continuous surjections.

For any $y = \langle F_\alpha(s) \rangle$ in Y and any pair $\alpha < \beta$, we have

$$p_\alpha(y) = F_\alpha(s) = f_{\alpha\beta}(F_\beta(s)) = f_{\alpha\beta}(p_\beta(y)),$$

so that $Y \subset X_\infty$. Since each X_α is compact Hausdorff, it follows that $\bar{Y}$ is a compact Hausdorff subspace of X_∞; thus there is a Cantor space K with closed subspace W and a continuous surjection $f\colon W \to \bar{Y}$. For each α define $f_\alpha\colon W \to X_\alpha$ by $f_\alpha = p_\alpha \circ f$, where p_α is the restriction to $\bar{Y}$ of the αth projection map in $\prod_\alpha X_\alpha$. Each f_α is a continuous surjection, being the composition of two such maps. In addition, for each pair $\alpha < \beta$,

$$f_\alpha = p_\alpha \circ f = (f_{\alpha\beta} \circ p_\beta) \circ f = f_{\alpha\beta} \circ (p_\beta \circ f) = f_{\alpha\beta} \circ p_\beta,$$

completing the proof.

In view of the above, we will say $\mathbb{S}$ *can be mapped onto* if condition (A) holds. The topological structure of the spaces that can map onto $\mathbb{S}$ will be determined by the structures of the $\{X_\alpha\}$ and the cardinality of the index set $\mathcal{A} = \{\alpha\}$.

III. **Specific types of inverse spectra.**

EXAMPLE. Let $\mathbb{S}$ be the inverse spectrum defined schematically in Figure 1, pointwise in Figure 2. Clearly $X_\infty = \varnothing$, so $\mathbb{S}$ cannot be mapped onto.

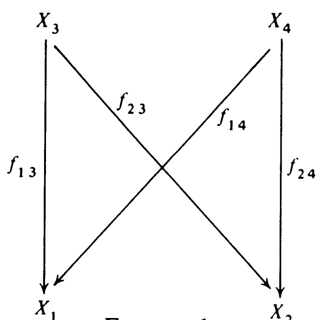

FIGURE 1

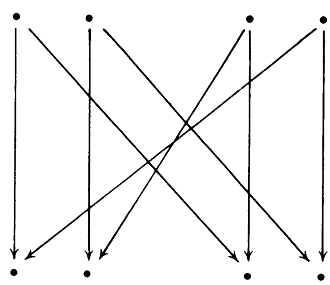

FIGURE 2

The difficulties with this inverse spectrum arise from the fact that the index set $\mathcal{A} = \{1, 2, 3, 4\}$ is not directed (no element of $\mathcal{A}$ follows both 3 and 4) but is partially directed (1 and 2 are unrelated, but each of 3 and 4 follows both). In the case of either extreme—the index set $\mathcal{A} = \{\alpha\}$ is directed, or $\mathcal{A}$ is not partially directed—we will show $\mathbb{S}$ can be mapped onto. The former case follows easily from known results. The latter is new, and requires some development.

THEOREM 2. *Let $\mathbb{S} = \{X_\alpha; f_{\alpha\beta}\}$ be an inverse spectrum of compact Hausdorff spaces where*

(a) *each $f_{\alpha\beta}$ is a continuous surjection,*

(b) *the index set $\mathcal{A} = \{\alpha\}$ is directed.*

Then $\mathbb{S}$ can be mapped onto.

PROOF. In view of (b) (see [1]),

$$p_\alpha(X_\infty) = \bigcap_{\beta > \alpha} (f_{\alpha\beta}(X_\beta)).$$

From (a), each map thus defined is surjective: thus the family $\{p_\alpha|_{X_\infty}\}$ maps X_∞ onto $\mathbb{S}$.

DEFINITION 2. Let $\mathbb{S}$ be an inverse spectrum, with $<$ the partial order on the index set $\mathcal{Q} = \{\alpha\}$. If

$$(*) \qquad \alpha \not< \beta \quad \text{and} \quad \beta < \gamma \text{ imply } \alpha \not< \gamma$$

we say $\mathbb{S}$ *is tree-like.* If in addition, $\mathcal{Q}$ has a unique minimal element, we say $\mathbb{S}$ *is a tree.*

Every tree-like inverse spectrum $\mathbb{S}$ can canonically be embedded in a tree $\mathfrak{T} = \mathfrak{T}(\mathbb{S})$. One need only adjoin a distinguished element $*$ to the index set $\mathcal{Q} = \{\alpha\}$ and set $* < \alpha$ for each $\alpha \in \mathcal{Q}$. Let $X_* = \{x_*\}$ be a one-point space, and set $f_{*\alpha}(x) = x_*$ for each x in each X_α. The spectrum $\mathfrak{T} = \{\{X_\alpha\} \cup X_*; \{f_{\alpha\beta}\} \cup \{f_{*\alpha}\}\}$ is a tree. Clearly $\mathbb{S}$ can be mapped onto if and only if $\mathfrak{T}$ can be mapped onto.

THEOREM 3. *Let $\mathfrak{T} = \{X_\alpha; f_{\alpha\beta}\}$ be a tree of compact Hausdorff spaces, where each $f_{\alpha\beta}$ is a continuous surjection. Then $\mathfrak{T}$ can be mapped onto.*

PROOF. We will show by transfinite induction that each $p_\alpha|_{X_\infty}: X_\infty \to X_\alpha$ is surjective; i.e., given $x_{\alpha_0} \in X_{\alpha_0}$ there is a point $y = \langle y_\alpha \rangle \in X_\infty$ with $y_{\alpha_0} = x_{\alpha_0}$.

Let $\ll$ be a well-ordering of $\mathcal{Q} = \{\alpha\}$, and denote by $P(\alpha)$ the proposition

$$(**) \qquad \begin{array}{l} y_\beta \text{ has been designated for all } \beta < \alpha; \\ \text{and if } \gamma < \beta < \alpha, \text{ then } f_{\gamma\beta}(y_\beta) = y_\gamma. \end{array}$$

Begin the induction by setting $y_{\alpha_0} = x_{\alpha_0}$ and designating $y_\gamma = f_{\gamma\alpha_0}(y_{\alpha_0})$ for all $\gamma < \alpha_0$. Now let α be given such that $P(\beta)$ holds for all $\beta \ll \alpha$. It may be that y_α has been designated: if $\alpha < \beta$ for some $\beta \ll \alpha$, then $(**)$ follows trivially. Suppose then that y_α has not been designated. One sees easily from $(*)$ of Definition 2 that

$$\mathfrak{B} = \{\beta \in \mathcal{Q} | \beta < \alpha \text{ and } y_\beta \text{ has been designated}\}$$

is totally ordered by $<$. If $\gamma, \beta \in \mathfrak{B}$ and $\gamma < \beta$, we have that $y_\gamma = f_{\gamma\beta}(y_\beta)$, so that

$$(***) \qquad \text{If } \gamma, \beta \in \mathfrak{B}, \quad \gamma < \beta \text{ implies } f_{\beta\alpha}^{-1}(y_\beta) \subset f_{\gamma\alpha}^{-1}(y_\gamma).$$

Each $f_{\beta\alpha}: X_\alpha \to X_\beta$ is a continuous surjection, so each $f_{\beta\alpha}^{-1}(y_\beta)$ is closed in X_α. By $(***)$, the collection $\{f_{\beta\alpha}^{-1}(y_\beta) | \beta \in \mathfrak{B}\}$ has the finite intersection property. Since X_α is compact Hausdorff, $\bigcap\{f_{\beta\alpha}^{-1}(y_\beta) | \beta \in \mathfrak{B}\}$ is nonempty. Designate any point in the intersection as y_α, and set $y_\beta = f_{\beta\alpha}(y_\alpha)$ for all $\beta < \alpha$. This completes the proof.

COROLLARY 2. *The following can be mapped onto.*

(1) *Any tree-like spectrum of compact Hausdorff spaces.*

(2) *Any spectrum of compact Hausdorff spaces where there is a set $\mathfrak{B} \subset \mathcal{Q}$ cofinal, where $\mathfrak{B}$ is tree-like.*

COROLLARY 3. *Let $\mathbb{S}$ be an inverse spectrum which, in addition to satisfying any of the conditions of Corollary* 2, *satisfies*

(i) *each X_α is compact metric,*

(ii) *the index set $\mathfrak{A} = \{\alpha\}$ is countable.*

Then the Cantor set 2^ω maps onto $\mathbb{S}$, with each f_α continuous. In particular, 2^ω maps continuously onto the inverse spectrum mentioned in the introduction.

The proofs of Corollaries 2 and 3 are obvious.

The author thanks the referee for suggesting he consider Theorem 1 in its full generality.

REFERENCE

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616

Current address: SESAME, % Physics, University of California, Berkeley, California 94720