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LINEAR GENERALIZATIONS OF GRONWALL'S INEQUALITY

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ABSTRACT. A variety of linear generalizations of Gronwall's inequality, including recent multivariable results of D. R. Snow and E. C. Young, are subsumed and extended by simple arguments involving the resolvent kernel of the integral operator.

"Everyone knows" that Gronwall's[1] inequality [5] is but one example of an inequality for a monotone operator $\mathcal{K}$ in which the exact solution of $w = a + \mathcal{K}w$ provides an upper bound on all solutions of $u \leqslant a + \mathcal{K}u$. Nevertheless, this idea is often neglected in deriving new variants of this classical inequality.

Here Gronwall's inequality is generalized to systems of n linear inequalities in m variables by arguments that amount to manipulation of the resolvent kernel equation for $\mathcal{K}$. These results encompass work of Chu and Metcalf [4], Snow [9], [10], Walter [11] (with a restriction noted below), Wendroff [1], and Young [14] as well as providing extensions to kernels having more general form and weaker regularity properties.

Let $G(x)$ and $H(x)$ denote real-valued $n \times n$ matrices and $a(x)$ and $u(x)$ denote n-vectors, all of which are continuous functions of $x = (x_1, \ldots, x_m)$. Let x^0 be a fixed m-vector and $\int_{x^0}^{x} dy$ denote the multiple integral $\int_{x_m^0}^{x_m} \cdots \int_{x_1^0}^{x_1} dy_1 \cdots dy_m$. Inequalities hold component-wise and I is the identity matrix.

THEOREM. *Let $G(x)$, $H(x)$ be continuous, nonnegative matrices for $x^0 \leqslant x$. If*

$$u(x) \leqslant a(x) + G(x)\int_{x^0}^{x} H(y)u(y)\,dy, \qquad x^0 \leqslant x, \tag{1}$$

then

$$u(x) \leqslant a(x) + G(x)\int_{x^0}^{x} V(x,y)H(y)a(y)\,dy, \qquad x^0 \leqslant x, \tag{2}$$

where $V(x, y)$ satisfies

$$V(x,y) = I + \int_{y}^{x} H(z)G(z)V(z,y)\,dz, \qquad x^0 \leqslant y \leqslant x. \tag{3}$$

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[1]Bellman [2], Reid [8], and Peano [7] deserve credit as well.

PROOF. In the norm in [11, pp. 141–142], the integral operator $\mathcal{K}$ on the right side of (1) is a contraction on the segment $x^0 \leqslant x \leqslant x'$ for any fixed x'. The resulting Neumann series consequently converges uniformly on any such compact set to a resolvent operator $(\mathcal{I} - \mathcal{K})^{-1}$, which is monotone because $\mathcal{K}$ is monotone.

A sharp bound on $u(x)$ is therefore the exact solution of $w = a + \mathcal{K}w$. The usual manipulations of the Neumann series, e.g. [13, pp. 147–149], show that this solution is just the right-hand side of (2), where $G(x)V(x, y)H(y)$ appears as the resolvent kernel of $G(x)H(y)$. The resolvent equation for $\mathcal{K}$ is (3) premultiplied by $G(x)$ and postmultiplied by $H(y)$; cf. [13, equation (37.9)]. $\square$

With $G(x) \equiv I$ and other restrictions, Snow [9], [10], Young [14], and Walter [11, pp. 143–144] have obtained inequalities like (2). Snow and Young regard $V(x, y)$ as the Riemann function for the initial-value problem equivalent to (3). (The equivalence of (3) and Snow's result [9] for a system of two inequalities follows from Snow's hypothesis that $H(y)$ is selfadjoint, which forces the same property on V.) Walter treats a more general region than $x^0 \leqslant x$, but he defines V via the Neumann series for the operator in (3).

COROLLARY 1. *Let $a(x) \geqslant 0$ and $G(x)$, $H(x) \geqslant 0$ for $x^0 \leqslant x$. Define*

$$J(z_1) \equiv \int_{y_2}^{x_2} \cdots \int_{y_m}^{x_m} H(z_1, z_2, \ldots, z_m) G(z_1, z_2, \ldots, z_m)\, dz_2 \cdots dz_m$$

and suppose that $J(z_1)$ commutes with $\exp \int_{y_1}^{z_1} J(s_1)\, ds_1$ for all $z_1 \geqslant y_1 \geqslant x_1^0$. If $u(x)$ satisfies (1), *then*

$$(4) \quad u(x) \leqslant a(x) + G(x) \int_{x^0}^{x} \exp\left(\int_y^x H(z)G(z)\, dz\right) H(y)a(y)\, dy, \quad x^0 \leqslant x.$$

PROOF. Let $E(z, y) \equiv \int_y^z H(s)G(s)\, ds$. Since $\exp E(x, y)$ is increasing in any component of its first argument, we have

$$\begin{aligned} \int_y^x H(z)G(z) \exp E(z, y)\, dz &\leqslant \int_{y_1}^{x_1} J(z_1) \exp\left(\int_{y_1}^{z_1} J(s_1) ds_1\right) dz_1 \\ &= \exp E(x, y) - I. \end{aligned}$$

Consequently, $\exp E(x, y)$ satisfies an integral inequality of which $V(x, y)$ is the exact solution in the case of equality; cf. (3). The fundamental argument of the theorem (that the solution of the equality provides a bound on all solutions of the corresponding inequality) now forces $V(x, y) \leqslant \exp E(x, y)$, and (4) follows from (2). $\square$

Corollary 1 extends a two-variable, scalar inequality originally due to Wendroff [1, p. 154, equation (30.2)].

In general, (4) is not sharp unless the inequalities depend only on a single *scalar* independent variable.

COROLLARY 2. *Let the vector $a(t)$ and the nonnegative matrices $G(t)$, $H(t)$ be functions of the single scalar variable t for $t^0 \leqslant t$. Assume that $H(t)G(t)$ and $\int_{t^0}^{t} H(s)G(s)\, ds$ commute for $t^0 \leqslant t$. If* (1) *holds (with t, t^0 in place of x, x^0), then*

$$u(t) \leqslant a(t) + G(t)\int_{t^0}^{t} \exp\left(\int_s^t H(r)G(r)\,dr\right) H(s)a(s)\,ds, \qquad t^0 \leqslant t.$$

PROOF. Integration reveals that (3) is satisfied by

$$V(t,s) = \exp \int_s^t H(r)G(r)\,dr. \quad \square$$

This corollary restates a result of Chu and Metcalf [4], which was obtained by summing a Neumann series, and it includes the classical inequalities of Gronwall *et al.* Willet's technique [12, Lemma 1] for treating kernels which are sums of terms like $G(t)H(s)$ could be used to solve (3) and thereby extend Corollary 2 to kernels of this more general form.

The commutativity assumptions in the preceding corollaries are imposed to permit integration of the matrix exponential function.

In the case of a scalar independent variable, Miller [6, pp. 189–201] has derived the resolvent kernel equations for a system of Volterra integral equations whose kernels are not necessarily continuous. The obvious extension of these results to several independent variables yields a substantially weakened form of the theorem. (The regularity condition given below is not the most general. See [6, pp. 200–201].)

ALTERNATE THEOREM. *Let $G(x)$, $H(x)$ be commuting, nonnegative matrices which are merely square integrable on $x^0 \leqslant x = (x_1, \ldots, x_m) \leqslant x'$ for each fixed $x' \geqslant x^0$. If* (1) *holds a.e. on $x^0 \leqslant x$, then* (2) *and* (3) *hold a.e. on $x^0 \leqslant x$.*

A differential analysis like that of Snow and Young obviously requires revision if the Riemann function V is defined by a differential equation whose coefficients may not be continuous. The integral equation approach taken here avoids this difficulty by requiring only enough smoothness in G and H to ensure that the resolvent kernel actually provides a solution of the integral equation.

The authors have also obtained multiple-index, discrete analogs of Gronwall's inequality by a similar approach. In the near future, they intend to unify both sets of results under an appropriate Volterra-Stieltjes integral equation formulation.

See [3] for a discussion in a similar spirit of nonlinear generalizations of Gronwall's inequality.

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