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COMPRESSIBLE MAPS

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ABSTRACT. Weingram has shown that if G is a finitely generated abelian group, then every nontrivial map $f\colon \Omega S^{2n+1} \to K(G, 2n)$ is incompressible; that is, f is not homotopic to a map whose image is contained in some finite-dimensional skeleton.

It is shown that a nontrivial map $\Omega S^{2n+1} \to K(G, 2n)$ may be compressible if G is not finitely generated. This result leads to some understanding of the obstructions to compressibility in Weingram's Theorem.

A map $f\colon X \to Y$ is said to compress into $A \subseteq Y$ if there is a map $f'\colon X \to A$ such that $f \simeq i \cdot f'$, where i is inclusion. Weingram [4] has shown that if G is a finitely generated abelian group, then every nontrivial map $f\colon \Omega S^{2n+1} \to K(G, 2n)$ is incompressible, that is, f is not homotopic to a map whose image is contained in some finite-dimensional skeleton.

It will be shown that a nontrivial map $\Omega S^{2n+1} \to K(G, 2n)$ may be compressible if G is not finitely generated. Specifically, if P is a set of primes and $Z_P \subseteq Q$ is the subgroup of P-local integers, then any map $\Omega S^{2n+1} \to K(Q/Z_P, 2n)$ compresses into the $(2n + 1)$-skeleton, provided $2 \notin P$ or $n = 1, 3$.

Let $M(G, k)$ denote a Moore space of type (G, k). For the remainder of this paper, it will be assumed that $2 \notin P$ or $2 \in P$ and $n = 1, 3$.

The following two well-known theorems will be stated without proof:

THEOREM (ADAMS [1]). *$M(Z_P, 2n + 1)$ is an H-space.*

THEOREM (STASHEFF [3]). *Let X and W be H-spaces and let $F\colon X \to W$ be an H-map. Let $\Omega W \to Y \to X$ be the fibration induced by f. Then Y is an H-space.*

LEMMA 1. *Any map $M(Z_P, 2n + 1) \to K(Q, 2n + 1)$ is homotopic to an H-map.*

PROOF. The obstructions to a map $f\colon X \to Y$ being homotopic to an H-map lie in $H^i(X \wedge X; \pi_i(Y))$. Since

$$\pi_i(K(Q, 2n + 1)) = 0 \quad \text{for } i \neq 2n + 1$$

and

$$H^{2n+1}(M(Z_P, 2n + 1) \wedge M(Z_P, 2n + 1); G) = 0 \quad \text{for any } G,$$

all obstructions lie in zero groups.

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In view of the preceding lemma, it will be assumed that any given map $f\colon M(Z_P, 2n+1) \to K(Q, 2n+1)$ is an H-map.

Let f be induced by the natural embedding $Z_P \subseteq Q$.

PROPOSITION 2. *Let $K(Q, 2n) \to E \to M(Z_P, 2n+1)$ be the fibration induced by f. Then E is an H-space.*

PROOF. This is an immediate consequence of the above two theorems and Lemma 1.

THEOREM 3.

$$\tilde{H}_i(E; Z) = \begin{cases} Q/Z_P, & i = 2n, \\ 0, & \textit{otherwise}. \end{cases}$$

PROOF. (Porter [2] has proved a similar theorem when the base space is S^3.) $H_*(K(Q, 2n); Z)$ is a polynomial algebra over Z on one generator in dimension $2n$ tensored with Q. Since f is induced by the natural embedding of $Z_P \subseteq Q$, the Serre exact sequence

$$H_{2n+1}(E) \to H_{2n+1}(M(Z_P, 2n+1)) \to H_{2n}(K(Q, 2n)) \to H_{2n}(E) \to 0$$

reduces to

$$0 \to Z_P \to Q \to H_{2n}(E) \to 0$$

and, hence, $H_{2n}(E) = Q/Z_P$.

In the homology spectral sequence, the only nonzero differential is d^{2n+1} which is an isomorphism for total degrees greater than $2n+1$ since $Z_P \otimes Q \cong Q$. Hence, $H_i(E) = 0$ for $i > 2n+1$.

REMARK. E is homotopic to a $(2n+1)$-dimensional Moore space

$$M(Q/Z_P, 2n).$$

PROPOSITION 4. *$K(Q/Z_P, 2n)$ is not homotopic to a finite-dimensional complex.*

PROOF. The exact sequence $0 \to Z_P \to Q \to Q/Z_P \to 0$ induces a fibration

$$K(Q/Z_P, 2n) \to K(Z_P, 2n+1) \to K(Q, 2n+1).$$

It is well known that $K(Q, 2n+1)$ is homotopic to a finite-dimensional complex. Hence if $K(Q/Z_P, 2n)$ were also homotopic to a finite-dimensional complex, it would imply that $H^k(K(Z_P, 2n+1); Z_p) = 0$ for all k greater than some integer. It suffices to show, then, that $H^k(K(Z_P, 2n+1); Z_p)$ is nonzero for infinitely many k, where $p \in P$.

Let ι be the fundamental class in $H^{2n+1}(K(Z, 2n+1); Z_p) = \operatorname{Hom}(Z, Z_p)$. It is well known that there are an infinite number of Steenrod operations $\mathcal{P}^I$ such that $\mathcal{P}^I \iota \neq 0$. ($I$ denotes an admissible sequence.) Hence it suffices to show that if $f\colon K(Z, 2n+1) \to K(Z_P, 2n+1)$ is the natural inclusion, then

$f^*(\bar{\iota}) = \iota$ for some $\bar{\iota} \in H^{2n+1}(K(Z_P, 2n+1); Z_p) = \mathrm{Hom}(Z_P, Z_p)$. But f^*: $\mathrm{Hom}\,(Z_P, Z_p) \to \mathrm{Hom}\,(Z, Z_p)$ is onto and the proposition is proved.

THEOREM 5. *Every nontrivial map $\Omega S^{2n+1} \to K(Q/Z_P, 2n)$ compresses into the $(2n+1)$-skeleton.*

PROOF. Any map f: $\Omega S^{2n+1} \to K(Q/Z_P, 2n)$ is homotopic to a map $r \cdot \Omega Sg$, where g: $S^{2n} \to K(Q/Z_P, 2n)$ and r is the retraction $\Omega SK(Q/Z_P, 2n) \to K(Q/Z_P, 2n)$.

But every such map g factors through $M(Q/Z_P, 2n)$ so that

$$S^{2n} \xrightarrow{g_1} M(Q/Z_P, 2n) \xrightarrow{g_2} K(Q/Z_P, 2n)$$

is nontrivial where

$$g \simeq g_2 \cdot g_1 .$$

By Theorem 3 and Proposition 2, $M(Q/Z_P, 2n)$ is an H-space so that g_1 extends to $\bar{g}_1$: $\Omega S^{2n+1} \to M(Q/Z_P, 2n)$. Then $f \simeq g_2 \cdot \bar{g}_1$, so f compresses into the $(2n+1)$-skeleton.

Observe that if $P = \{p\}$, then $Q/Z_P = Z_{p^\infty} = \varinjlim_r Z_{p^r}$ induced by the inclusion $Z_{p^r} \to Z_{p^{r+1}}$. Let G be a finitely-generated odd torsion group so that $G = \oplus_{i=1}^m Z_{p_i^{\alpha_i}}$. Let $P = \{p_1, \ldots, p_m\}$ and $G_{k+1} = \oplus_{i=1}^m Z_{p_i^{\alpha_i+k}}$. Then $Q/Z_P \cong \varinjlim_k G_k$.

Since $\pi_{2n+1}(M(G_r, 2n)) = G_r \otimes Z_2 = 0$, and $M(G_r, 2n)$ can be thought of as the $(2n+1)$-skeleton of $K(G_r, 2n)$, the obstructions to extending

$$S^{2n} \to K(G_r, 2n)^{(2n+1)}$$

to $\Omega S^{2n+1} \to K(G_r, 2n)^{(2n+1)}$ are the obstructions to extending

$$S^{2n} \to M(G_r, 2n)$$

to $\Omega S^{2n+1} \to M(G_r, 2n)$.

Let $j_{r,m}$: $M(G_r, 2n) \to M(G_{r+m}, 2n)$ be inclusion and let j: $S^{2n} \to M(G_r, 2n)$.

THEOREM 6. *For every k and r, there is an m such that $j_{r,m} \cdot j$ extends to $(S^{2n})_k$, the kth reduced product of S^{2n}.*

PROOF. Let $j_{r,\infty}$: $M(G_r, 2n) \to M(Q/Z_P, 2n)$ be the inclusion. Since $M(Q/Z_P, 2n)$ is an H-space, it follows that $j_{r,\infty} \cdot j$ extends to $(S^{2n})_k$ for all k. Let $\bar{j}$ be such an extension. Since $(S^{2n})_k$ is compact, image$\bar{j} \subseteq M(G_{r+m}, 2n)$ for some m and, hence, $\bar{j}$ is an extension of $j_{r,m} \cdot j$.

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