PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, October 1976

SOME NOWHERE EQUICONTINUOUS HOMEOMORPHISMS[1]

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ABSTRACT. It is shown that a nowhere equicontinuous homeomorphism can be defined on a compact polyhedron X if and only if X does not have cell decomposition which contains a principal 1-cell. It is also shown that for each locally connected contractible continuum C in the plane, there is a nowhere equicontinuous homeomorphism h_c on a disk in the plane such that the fixed point set of h_c is C.

1. **Introduction.** Let X be a metric space with a metric d and h a homeomorphism on X (a homeomorphism of X onto itself). We say that h is *equicontinuous* at $x \in X$ if $\{h^n | n \in Z\}$ is an equicontinuous family at x. The set $\{x \in X | h$ is equicontinuous at $x\}$ is called the *regular set* of h and its complement in X is called the *irregular set* of h. If the regular set of h is empty, we say that h is a *nowhere equicontinuous homeomorphism* (NEH). Homeomorphisms h_1 and h_2 on X are said to be *topologically equivalent* if there is a homeomorphism k on X such that $h_1 = k^{-1} h_2 k$.

It is a known fact that neither the closed unit interval nor the circle admit a NEH [2]. In fact, it is true that if h: $X \to X$ is a homeomorphism on X, where X is either the closed unit interval or the circle, then the irregular set of h is nowhere dense in X. It can be easily shown that, for compact spaces, the property of admitting a NEH is topological. In §2 of this paper we show that a compact polyhedron admits a NEH if and only if it does not have a cell decomposition which contains a principal 1-cell (Theorem 5). We also show that for each locally connected contractible continuum C in the interior of the unit disk, with diam $(C) > 0$, there is a NEH h^* on the unit disk such that Fix $(h^*) = C$ where Fix $(h^*) = \{x | h^*(x) = x\}$ (Theorem 7).

Throughout this paper we use such standard terminologies as orbit, dense orbit, periodic point and refer readers to [1] for definitions. We also use some standard terminologies of piecewise linear topology and refer readers to [4] for their definitions. The symbols I, B^n, S^n, $\mathring{X}$ and ∂X are used to denote the closed unit interval [0, 1], the n-ball, the n-sphere, combinatorial interior of X

Received by the editors October 6, 1975 and, in revised form, December 15, 1975.

AMS (*MOS*) *subject classifications* (1970). Primary 57E20, 54H20.

Key words and phrases. Equicontinuous homeomorphism, irregular set, dense orbit, principal n-cell, strong cellularity.

[1] This paper will constitute a part of the author's Ph.D. Thesis, written at Oklahoma State University under the direction of Professor P. F. Duvall.

and the combinatorial boundary of X respectively. A *principal n-cell* in a complex is an n-cell which intersects higher dimensional cells in a subset of its boundary. All spaces considered here are metric spaces and a map is a continuous function.

The author wishes to express his gratitude to Professor P. F. Duvall, Jr. for helpful suggestions.

2. **The existence of NEH's.** It can be easily shown that the homeomorphism h on S^1, the unit circle in the complex plane, defined by $h(e^{i\theta}) = e^{i(\theta+2\pi t)}$, $0 \leqslant t \leqslant 1$, is periodic with the period q if t is rational and $t = P/q$ in the lowest term, and each point of S^1 has dense orbit in S^1 under h if t is irrational. Now we proceed with the construction of a NEH on some compact polyhedron.

LEMMA 1. *Define h: $S^1 \times I \to S^1 \times I$ by $h(e^{i\theta}, t) = (e^{i(\theta+2\pi t)}, t)$. Then h is a NEH on $S^1 \times I$.*

PROOF. Assume that the metric d on $S^1 \times I$ is the product metric. Let $x \in S^1 \times I$ and write $x = (e^{i\theta}, t)$, $0 \leqslant \theta \leqslant 2\pi$, $t \in I$. Take $\varepsilon = \sqrt{2}$. For each δ, take t' such that $0 < |t - t'| < \min\{\delta, \frac{1}{4}\}$. Then $0 < d((e^{i\theta}, t), (e^{i\theta}, t')) < \delta$ and there is an integer n such that $\frac{1}{4} \leqslant n(t - t') \leqslant \frac{1}{2}$. Thus

$$
\begin{aligned}
d(h^n(e^{i\theta}, t), h^n(e^{i\theta}, t')) &= d((e^{i(\theta+2n\pi t)}, t), (e^{i(\theta+2n\pi t')}, t')) \\
&> d((e^{i(\theta+2n\pi t)}, t), (e^{i(\theta+2n\pi t')}, t)) = |e^{i(\theta+2n\pi t)} - e^{i(\theta+2n\pi t')}| \\
&= 2 \sin n\pi(t - t') \geqslant \sqrt{2}
\end{aligned}
$$

by the choice of n.

COROLLARY 2. *Let X be a space. If there is a map f: $X \to I$ such that* Int $(f^{-1}(t))$, *the point set interior of $f^{-1}(t)$, is empty, for each $t \in I$, then $S^1 \times X$ admits a NEH.*

PROOF. Assume that $S^1 \times X$ has the product metric. Define g: $S^1 \times X \to S^1 \times X$ by $g((e^{i\theta}, x)) = (e^{i(\theta+\pi \cdot f(x))}, x)$. Take $\varepsilon = \sqrt{2}$. For each $(e^{i\theta}, x) \in S^1 \times X$ and any neighborhood U of $(e^{i\theta}, x)$, there is a $\delta > 0$ such that the δ-neighborhood $N_\delta(x)$ of x is contained in $\pi_x(U)$, where π_x is the projection map of $S^1 \times X$ onto X. Thus, there is a point $y \neq x$ in $N_\delta(x)$ such that $f(x) \neq f(y)$, since Int $(f^{-1}(t)) = \varnothing$ for each t. Therefore $0 < |f(x) - f(y)| \leqslant 1$ so that $\frac{1}{2} < n(f(x) - f(y)) \leqslant 1$ for some integer n. Then

$$
\begin{aligned}
d(h^n(e^{i\theta}, x), h^n(e^{i\theta}, y)) &= d((e^{i(\theta+n\pi \cdot f(x))}, x), (e^{i(\theta+n\pi \cdot f(y))}, y)) \\
&> d((e^{i(\theta+n\pi f(x))}, x), (e^{i(\theta+n\pi f(y))}, x)) \geqslant \sqrt{2}.
\end{aligned}
$$

LEMMA 3. *There is a NEH ζ_2 on B^2 such that $\zeta_2|_{\partial B^2}$, the restriction of ζ_2 to ∂B^2, is $1_{\partial B^2}$, the identity map of ∂B^2.*

PROOF. Let f: $S^1 \times I \to B^2$ be a map which satisfies the following conditions: $f|_{S^1\times(0,1]}$ is a homeomorphism of $S^1\times(0,1]$ onto $B^2-\{(x,0)|-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}\}$, $f((e^{i\theta},0))=f((e^{i(2\pi-\theta)})), 0\leqslant\theta\leqslant 2\pi$, $f((e^{i\pi},0))=(-\frac{1}{2},0)$, $f((e^{i\theta},0))=(\frac{1}{2},0)$ and $f|_{\{(e^{i\theta},0)|0\leqslant\theta\leqslant\pi\}}$ is a homeomorphism of $\{(e^{i\theta},0)|0\leqslant\theta\leqslant\pi\}$ onto $\{(x,0)|-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}\}$. Take $h\colon S^1\times I\to S^1\times I$ defined in Lemma 1. Define $\zeta_2\colon B^2\to B^2$ by $\zeta_2(x)=fhf^{-1}(x)$. Then it is easy to see that ζ_2 is a homeomorphism on B^2. If $p\in B^2-\{(x,0)|-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}\}$ then $p=f((e^{i\theta},t))$ for some $t\neq 0$. Since $f|_{S^1\times[t/2,1]}$ is a homeomorphism of $S^1\times[t/2,1]$ onto an annulus $A\subset B^2-[-\frac{1}{2},\frac{1}{2}]$, both $f|_{S^1\times[t/2,1]}$ and $f^{-1}|_A$ are uniformly continuous. Thus, if $\zeta_2|_A$ were equicontinuous at p then $h|_{S^1\times[t/2,1]}$ would be equicontinuous at $(e^{i\theta},t)$. Therefore $\zeta_2|_A$ is not equicontinuous at p so that ζ_2 is not equicontinuous at p. If $p\in\{(x,0)|-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}\}$, then $\zeta_2(p)=p$. Choose $\varepsilon>0$ so that $N_{2\varepsilon}(p)$ does not contain $\{(x,0)|-\frac{1}{2}\leqslant x\leqslant\frac{1}{2}\}$. Then there is an η and a neighborhood U of $e^{i\eta}$ such that $U\times[0,t]\subset S^1\times[0,t]-f^{-1}(N_\varepsilon(p))$ for each $t\in I$. For each $\delta>0$, pick $(e^{i\theta},t)\in f^{-1}(N_\delta(p))$ which has dense orbit in $S^1\times\{t\}$ under h. Then there is an integer n such that $h^n((e^{i\theta},t))\in U\times\{t\}$. Therefore $\zeta_2^n(f((e^{i\theta},t)))\notin N_\varepsilon(p)$ and $f((e^{i\theta},t))\in N_\delta(p)$ which shows that ζ_2 is not equicontinuous at p. It is clear that $\zeta_2|_{\partial B^2}=1_{\partial B^2}$.

LEMMA 4. *For each $n\geqslant 2$, B^n admits a NEH ζ_n such that $\zeta_n|_{\partial B^n}=1_{\partial B^n}$.*

PROOF. We prove this lemma by induction on n. By Lemma 3, B^2 admits such a homeomorphism. Assume that there is such a homeomorphism ζ_{n-1} on B^{n-1}. For each θ, $0\leqslant\theta<2\pi$, let

$$B_\theta^{n-1}=\left\{(x_1,\ldots,x_{n-2},x_{n-1}\cos\theta,x_{n-1}\sin\theta)\,|\,\sum_{i=1}^{n-1}x_i^2\leqslant 1\text{ and }x_{n-1}\geqslant 0\right\}.$$

Then B^{n-1} is the closed half of the unit ball sitting in the subvector space in R^n of dimension $n-1$ which is determined by R^{n-2} and the vector $(0,\ldots,0,\cos\theta,\sin\theta)\in R^n$. Thus it is easy to see that

$$B^n=\bigcup_{0\leqslant\theta<2\pi}B_\theta^{n-1}\quad\text{and}\quad B_\theta^{n-1}\cap B_{\theta'}^{n-1}=B^{n-2}$$

for $\theta\neq\theta'$. Since $(B^{n-1},\partial B^{n-1})$ and $(B_0^{n-1},\partial B_0^{n-1})$ are homeomorphic as compact pairs, there is a NEH $\psi_{n-1}\colon B_0^{n-1}\to B_0^{n-1}$ such that $\psi_{n-1}|_{\partial B_0^{n-1}}=1_{\partial B_0^{n-1}}$. Define $\zeta_n\colon B^n\to B^n$ by $\zeta_n|_{B_\theta^{n-1}}=\rho_\theta\psi_{n-1}\rho_\theta^{-1}$ where $\rho_\theta\colon B_0^{n-1}\to B_\theta^{n-1}$ is the homeomorphism defined by

$$\rho_\theta((x_1,\ldots,x_{n-2},x_{n-1},0))=(x_1,\ldots,x_{n-2},x_{n-1}\cos\theta,x_{n-1}\sin\theta).$$

Then ζ_n is a well-defined function since $\rho_\theta\psi_{n-1}\rho_\theta^{-1}|_{B^{n-2}}=1|_{B^{n-2}}$ for any θ. Let $x\in B^n-B^{n-2}$. Then a sequence

$$\{x^i=(x_1^i,\ldots,x_{n-2}^i,x_{n-1}^i\cos\theta^i,x_{n-1}\sin\theta^i)\}_{i=1}^\infty$$

converges to

$$x = (x_1, \ldots, x_{n-2}, x_{n-1} \cos \theta, x_{n-1} \sin \theta)$$

if and only if $\{(x_1^i, \ldots, x_{n-2}^i, x_{n-1}^i)\}_{i=1}^{\infty}$ converges to $(x_1, \ldots, x_{n-2}, x_{n-1})$ and $\{\theta^i\}_{i=1}^{\infty}$ converges to θ up to modulo 2π. Thus, the continuity of ζ_n at $x \in B^n - B^{n-2}$ is clear. Suppose $x \in B^{n-2}$. Then a sequence $\{x^i\}_{i=1}^{\infty}$ converges to x if and only if $\{\rho_\theta^{-1} i(x^i)\}_{i=1}^{\infty}$ converges to x, since $d(x, \rho_\theta^{-1} i(x^i)) = d(x, x^i)$ for each i. Therefore ζ_n is continuous at x. Since the map $\zeta'_n\colon B^n \to B^n$ defined by $\zeta'_n|_{B_\theta^{n-1}} = \rho_\theta \psi_{n-1}^{-1} \rho_\theta^{-1}$ is the inverse of ζ_n, ζ_n is a homeomorphism. Furthermore, $\zeta_n|_{B_\theta^{n-1}}$ is the identity on ∂B^{n-1} for each θ so that $\zeta_n|_{\partial B^n} = 1_{\partial B^n}$ since $\partial B^n \subset \bigcup_{0<\theta<2\pi} \partial B_\theta^{n-1}$. ζ_n is NEH since $\zeta_n|_{B_\theta^{n-1}}$ is NEH for each θ.

THEOREM 5. *A compact polyhedron P admits a NEH if and only if P contains no principal* 1*-cells.*

PROOF. To prove the necessity, suppose that P contains a principal 1-cell and suppose that there is a NEH h on P. Let K be a triangulation of P, K_1 the collection of principal 1-cells in K and write $|K_1| = P_1$. Then $h(P_1) = P_1$. Since $P_1 \cap |K - K_1|$ is finite, the regular set of $h|_{P_1}$ is at most finite. But by using the fact that the irregular set of a homeomorphism on either a circle or a closed 1-cell is nowhere dense, we can show that the irregular set of a homeomorphism on P_1 is nowhere dense. Therefore h cannot be a NEH on P.

If P does not contain any principal 1-cell, then we can write $P = \bigcup \{\sigma_j\}_{j=1}^{k}$ where σ_j is a principal n-simplex with $n \geqslant 2$ in some triangulation $\{\sigma_i\}_{i=1}^{m}$. Therefore, since $\zeta_n|_{\partial B^n} = 1_{\partial B^n}$, we can define a NEH h on P by taking h to be $g_n \zeta_n g_n^{-1}$ on each principal cell σ_j of dimension n where $g_n\colon B^n \to \sigma_j$ is a homeomorphism of B^n onto σ_j.

LEMMA 6. *Let C be a locally connected contractible continuum in* Int (B^2), *where $B^2 \subset R^2$. If C is nowhere dense in R^2, then there is a map f from S^1 onto C such that the pair (M_f, C) is homeomorphic to (B^2, C) where M_f denotes the mapping cylinder associated with f.*

PROOF. Since C is strongly cellular [5], there is a circle S and a homotopy H of S in R^2 such that

(1) H_0 is the identity,

(2) H_t is an embedding for $t < 1$,

(3) $H_t(S) \cap H_u(S) = \varnothing$ for $t \neq u$, and

(4) $h_1(S) = C$ [3, Theorem 2.1].

By the Schoenflies theorem, S bounds a disk. Therefore we may assume that $S = S^1$. It is clear, from the properties of H, that $H|_{S^1 \times [0,1)}$ is an embedding and $\operatorname{Im}(H) \subset B^2$. To prove that $\operatorname{Im}(H) = B^2$, suppose that there is $x \in \operatorname{Int}(B^2) - C$ such that $\operatorname{Im}(H) \subset B^2 - \{x\}$. Then there is a retraction $\gamma\colon B^2 - \{x\} \to S^1$. Now, γH_1 is homotopic to $\gamma H_0 = 1_{S^1}$. But, since C is contractible, γH_1 is null homotopic. Thus, we obtain a contradiction. By taking $f = H_1$, we see that (M_f, C) is homeomorphic to (B^2, C).

THEOREM 7. *For each locally connected contractible continuum* C *which is nowhere dense in* $\operatorname{Int}(B^2)$ *with* $\operatorname{diam}(C) > 0$, *there is a NEH* h_C *on* B^2 *such that* $\operatorname{Fix}(h_C) = \{x \in B^2 | h_C(x) = x\}$ *is* C.

PROOF. Let f be the map in Lemma 6. Since f is a closed map from S onto C, C has the identification topology with respect to f. Thus, (M_f, C) is homeomorphic to $(S^1 \times I/\sim, \{[e^{i\theta}, 1] | e^{i\theta} \in S^1\})$ where $\sim$ is the equivalence relation on $S^1 \times I$ induced by the map H which is defined in Lemma 6 and $[x, t]$ denotes the equivalence class of (x, t). Write $\{[e^{i\theta}, 1] | e^{i\theta} \in S^1\} = C'$. Then it suffices to show the existence of NEH h^* on $S^1 \times I/\sim$ with $\operatorname{Fix}(h^*) = C'$. Let $p: S^1 \times I \to S^1 \times I/\sim$ be the projection and $h: S^1 \times I \to S^1 \times I$ be defined by $h(e^{i\theta}, t) = (e^{i(\theta + \pi(1-t))}, t)$. Then, by the argument used in Lemma 1, h is a NEH on $S^1 \times I$ and $\operatorname{Fix}(h) = S^1 \times \{1\}$. Define $h^*: S^1 \times I/\sim \to S^1 \times I/\sim$ by $h^*([e^{i\theta}, t]) = ph(e^{i\theta}, t)$. Then, since h^* is a well defined one-to-one correspondence, it is a homeomorphism. Since $p|_{S^1 \times [0,t]}$ is a homeomorphism and $S^1 \times [0, t]$ is compact for each $t < 1$, it is clear that $\{[e^{i\theta}, s] \in S^1 \times I/\sim \,|\, s < 1\} \subset \operatorname{Irr}(h^*)$. To show that $[e^{i\theta}, 1] \in \operatorname{Irr}(h^*)$, note first that $\operatorname{Fix}(h^*) = C'$ and $\operatorname{diam}(C') > 0$. For each neighborhood U of $[e^{i\theta}, 1]$, U contains $[e^{i\theta}, t]$ for some irrational t. Since the orbit of $(e^{i\theta}, t)$ under h is dense in $S^1 \times \{t\}$, the orbit of $[e^{i\theta}, t]$ under h^* is dense in $\{[e^{i\theta}, t] | 0 \leqslant \theta \leqslant 2\pi\}$. Now, if we take $\delta = \frac{1}{3}\operatorname{diam}(C')$, then we can find $n \in Z$ such that $d(h^{*n}[e^{i\theta}, 1], h^{*n}[e^{i\theta}, t]) > \delta$.

If h_1 and h_2 are topologically equivalent homeomorphisms then $\operatorname{Fix}(h_1)$ is homeomorphic to $\operatorname{Fix}(h_2)$. Consequently Theorem 7 implies the existence of uncountably many conjugacy classes of nowhere equicontinuous homeomorphisms.

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