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PARTIAL DIFFERENTIAL EQUATIONS ON SEMISIMPLE LIE GROUPS

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ABSTRACT. Suppose G is a noncompact, connected, semisimple Lie group with finite center and K is a maximal compact subgroup. Let D be an Ad K-invariant element in the complexified enveloping algebra of G. The main result of this paper gives criterion for when the map $D: \mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective, where $\mathcal{E}'(G)$ is the space of compactly supported distributions on G.

1. **Introduction.** Let G be a noncompact, connected, semisimple, Lie group with finite center. Fix $G = KAN$, an Iwasawa decomposition of G. That is, K is a maximal compact subgroup of G, A is a vector subgroup of G with $\operatorname{Ad} A$ consisting of semisimple transformations and A normalizes N, a simply connected nilpotent subgroup of G. Let M be the centralizer of A in K. We denote the Lie algebras of G, K, M, A, and N by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$ respectively.

We use the notation $\mathcal{E}(G) = C^\infty(G)$ and $\mathcal{D}(G) = C_0^\infty(G)$ and denote their respective duals by $\mathcal{E}'(G)$ and $\mathcal{D}'(G)$. Left and right translation by elements of G induce linear maps on these spaces as follows.

For $x, g \in G$ and f in either $\mathcal{D}(G)$ or $\mathcal{E}(G)$,

$$L(g)f(x) = f(g^{-1}x), \qquad R(g)f(x) = f(xg),$$

and for T in the dual of f,

$$\langle L(g)T, f\rangle = \langle T, L(g^{-1})f\rangle, \qquad \langle R(g)T, f\rangle = \langle T, R(g^{-1})f\rangle$$

($\langle , \rangle$ is complex bilinear).

Let $\mathcal{E}(G: F)$ ($\mathcal{D}'(G: F)$) denote the C^∞ functions (distributions) on G which are left and right K-finite and set

$$\mathcal{E}'(G: F) = \mathcal{E}'(G) \cap \mathcal{D}'(G: F), \qquad \mathcal{D}(G: F) = \mathcal{E}(G: F) \cap \mathcal{D}(G).$$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$, let $U(\mathfrak{g})^{\mathfrak{k}}$ be the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$ and $\mathfrak{Z}$ the center of $U(\mathfrak{g})$. The main purpose of this paper is to give a criterion (Lemma 3.1) for when $D(\mathcal{E}(G))$ is dense in $\mathcal{E}(G)$ for $D \in U(\mathfrak{g})^{\mathfrak{k}}$ and to apply this criterion to elements of $\mathfrak{Z}$. In this paper we only prove that this criterion is sufficient, but in a subsequent paper [6], we show that this criterion is necessary for linear noncompact semisimple Lie groups.

Finally, we apply our techniques to the study of invariant differential

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operators on homogeneous vector bundles. In this paper we do not need to prove an analog of the classical Paley-Wiener theorem but rest our main techniques on Harish-Chandra's characterization of cusp forms and his Fourier expansions of C^∞-functions on G.

2. **Principal series and the Fourier transform.** We first recall the definition of the principal series of G. Let ω: $M \to Gl(H)$ be an irreducible unitary representation of M and let $\nu \in \mathfrak{a}_c^*$. ω and ν define a representation $\nabla_{\omega,\nu}$ of the group $MAN = P_0$ on H by setting $\nabla_{\omega,\nu}(man) = e^{(i\nu+\rho)(\log a)}\omega(m)$ ($m \in M, a \in A, n \in N$) where, for $H \in \mathfrak{a}$, $2\rho(H) = \operatorname{tr} \operatorname{ad} H|_{\mathfrak{n}}$. Now let $H^{\omega,\nu}$ be the set of all measurable functions f: $G \to H$ such that:

(1) $f(gp) = \nabla_{\omega,\nu}(p)^{-1}f(g)$ ($g \in G, p \in P_0$); and

(2) $\int_K \|f(k)\|^2 dk = \|f\|^2 < \infty$.

Observe that we have an inner product on $H^{\omega,\nu}$, $(u,v) = \int_K (u(k), v(k))\, dk$ ($u, v \in H^{\omega,\nu}$), and this turns $H^{\omega,\nu}$ into a Hilbert space. Now left translation induces a representation $\Pi_{\omega,\nu}$ of G on $H^{\omega,\nu}$. Let $X^{\omega,\nu}$ denote the set of all K-finite vectors of $H^{\omega,\nu}$ and observe that as a K-module, $X^{\omega,\nu}$ is isomorphic to the space

$$X(\omega) = \{u\colon K \to H \,|\, u \text{ is left } K\text{-finite and } u(km) = \omega(m)^{-1}u(k) \quad \text{for all } k \in K, m \in M\}.$$

. We will abuse notation throughout and identify $X^{\omega,\nu}$ with $X(\omega)$ and any distinction will be carried by the term $\Pi_{\omega,\nu}$. Observe now that $\Pi_{\omega,\nu}$ induces a representation of the algebra $U(\mathfrak{g})$ on $X(\omega)$.

If $f \in D(G)$ we define, as usual,

$$\Pi_{\omega,\nu}(f) = \int_G f(x)\Pi_{\omega,\nu}(x)\, dx.$$

If $T \in \mathcal{E}'(G)$ we may attempt to extend the above definition by setting $(\Pi_{\omega,\nu}(T)u, v) = \langle T, h\rangle$, where u, $v \in H^{\omega,\nu}$ and $(\Pi_{\omega,\nu}(x)u, v) = h(x)$. However, as $h(x)$ is not necessarily C^∞, we see that for a general such T, $\Pi_{\omega,\nu}(T)$ is only densely defined. Suppose now that $T \in \mathcal{E}'(G\colon F)$ and define, for u, $v \in X(\omega)$, $(\Pi_{\omega,\nu}(T)u, v)$ as above. Note that $\Pi_{\omega,\nu}(T)$ restricted to $X(\omega)$ is well defined and that $(\Pi_{\omega,\nu}(T)u, v) = 0$ if u,v are not in a minimal $\Pi_{\omega,\nu}(T)$ invariant space, which we denote as $X(\omega, T)$. As the closure of $X(\omega, T)$ in $H^{\omega,\nu}$ is $X(\omega, T)$, $\Pi_{\omega,\nu}(T)$ extends to a linear operator on $H^{\omega,\nu}$.

LEMMA 2.1. *Fixing $\omega \in \hat{M}$ we have that the map $\mathfrak{a}_c^* \to$ End $X(\omega, T) \subseteq$ End $H^{\omega,\nu}$, given by $\nu \to \Pi_{\omega,\nu}(T)$, is holomorphic.*

PROOF. Now $\Pi_{\omega,\nu}(T)$ has finite rank, and it suffices to show that for a fixed u, $v \in X(\omega)$ the map $\nu \to (\Pi_{\omega,\nu}(T)u, v)$ is holomorphic. Now set $f_\nu(x) = (\Pi_{\omega,\nu}(x)u, v)$, and we have that $(\Pi_{\omega,\nu}(T)u, v) = \langle T, f_\nu\rangle$, where

$$f_\nu(x) = \int_K \exp\{-(i\nu + \rho)H(\rho^{-1}k)\}(u(K(x^{-1}k), v(k)))\, dk,$$

where for $g \in G$, $g = K(g)\exp H(g)n(g)$ ($K(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$). As $T \in \mathcal{E}'(G)$ we have that there exists a compact set $V \subseteq G$ such that

$\operatorname{supp} T \subset V$ and $KVK = V$. Now there exist $D_1, \ldots, D_n \in U(\mathfrak{g})$ and $C > 0$ such that for $f \in \mathcal{D}(G)$,

$$|\langle T, f \rangle| \leq C \sum_{i=1}^{n} \sup_{x \in V} |D_i f(x)|.$$

For $e \in \mathfrak{a}_{\mathbf{c}}^*$ and for W a compact set with $\operatorname{int} W \supset V$, we see that $1/\xi(f_{\nu+\xi e}(x) - f_\nu(x))$ converges uniformly on W as $\xi \to 0$ to

$$F_\nu(x) = \int_K (-i)e(H(x^{-1}k))e^{-(i\nu+\rho)(H(x^{-1}k))}(U(K(x^{-1}k)), v(k))\,dk$$

and if $D \in U(\mathfrak{g})$, $1/\xi(Df_{\nu+\xi e}(x) - Df_\nu(x))$ converges uniformly to $DF_\nu(x)$ on W by the Leibniz formula. Thus $\langle T, 1/\xi(f_{\nu+\xi e} - f_\nu)\rangle$ converges to $\langle T, F_\nu\rangle$ as $\xi \to 0$ and, hence, $\nu \to (\Pi_{\omega,\nu}(T)u, v)$ is holomorphic.

The map $\nu \to \Pi_{\omega,\nu}(T)$ also satisfies growth conditions which we will not examine here.

LEMMA 2.2. *Suppose $T \in \mathcal{E}'(G\colon F)$ and $\Pi_{\omega,\nu}(T) = 0$ for all $\omega \in \hat{M}$ and all $\nu \in \mathfrak{a}_{\mathbf{c}}^*$. Then $T = 0$.*

PROOF. Let $g \in D(G)$ and consider the function $g_* T \in \mathcal{D}(G)$ where $g_* T(x) = \langle T, L_{x^{-1}}\tilde{g}\rangle$ with $\tilde{g}(y) = g(y^{-1})$. An elementary calculation then yields $\Pi_{\omega,\nu}(g_* T) = \Pi_{\omega,\nu}(g)\Pi_{\omega,\nu}(T)$. Thus, setting $f = g_* T$, we have that $\Pi_{\omega,\nu}(f) = 0$ for all $\omega \in \hat{M}$ and all $\nu \in \mathfrak{a}_{\mathbf{c}}^*$. We now examine $\Pi_{\omega,\nu}(f)$ more closely.

Let $h\colon K \to H$ be a continuous function such that $h(km) = \omega(m)^{-1}h(k)$ for $k \in K$, $m \in M$. Then, for any $\nu \in \mathfrak{a}_{\mathbf{c}}^*$, h extends to a function in $H^{\omega,\nu}$. Now

$$\Pi_{\omega,\nu}(f) = \int_G f(x)h(x^{-1}k)\,dk = \int_K K_{f,\nu}(k,k')h(k')\,dk'$$

where

$$K_{f,\nu}(k,k') = \int_A da \int_N dn \int_M dm\ \omega(m)f(kmank'^{-1})e^{-(i\nu+\rho)(\log a)}.$$

As $f \in \mathcal{D}(G)$, we see that $K_{f,\nu}(k,k')$ is continuous, and since $\Pi_{\omega,\nu}(f) = 0$ for all $\omega \in \hat{M}$, we know that $K_{f,\nu} = 0$ for all $\nu \in \mathfrak{a}_{\mathbf{c}}^*$. From the Plancherel formula for $\mathbf{R}^n$ we see that

$$\int_N dn\, f(kank'^{-1}) = 0 \qquad (k, k' \in K, a \in A).$$

Now let U be the unipotent radical of a minimal parabolic subgroup of G and consider the function

$$F(g) = \int_U f(gu)\,du.$$

As is well known there is a $k' \in K$ such that $U = k'Nk'^{-1}$. Thus

$$F(g) = \int_N f(gk'nk'^{-1})\,dn.$$

Now writing $gk' = kan_1$ $(k \in K, a \in A, n_1 \in N)$ we see that

$$F(g) = \int_N f(kank'^{-1})\,dn = 0$$

for any $g \in G$. We now show that $f = 0$.

Let P_1 be a parabolic subgroup of G which is minimal with respect to the property that

$$f_{P_1}(g) = \int_{U_1} f(gu)\,du \not\equiv 0,$$

where U_1 is the unipotent radical of P_1. Since f is the (possibly infinite) sum of functions in $\mathfrak{D}(G\colon F)$ (see Harish-Chandra [3]), to show that $f = 0$ it suffices to prove $f = 0$ under the assumption that $f \in \mathfrak{D}(G\colon F)$. Now $P_1 = M_1 U_1$, where M_1 is reductive, and thus we have that f_{P_1} is a cusp form on M_1 with compact support. From Harish-Chandra [2] we see that $f_{P_1} = 0$ since P_1 is not conjugate to P_0, which is a contradiction. Therefore, $f = 0$.

As $g_* T = 0$ for any $g \in \mathfrak{D}(G)$, we have that $T = 0$.

LEMMA 2.3. *For $D \in U(\mathfrak{g})^{\mathfrak{k}}$, $D\colon \mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective if and only if $D\colon \mathcal{E}'(G\colon F) \to \mathcal{E}'(G\colon F)$ is injective.*

PROOF. The only if part is obvious. Suppose now that $D\colon \mathcal{E}'(G\colon F) \to \mathcal{E}'(G\colon F)$ is injective. Let $f \in \mathcal{E}(G)$ and observe from Harish-Chandra [3] that

$$f = \sum_{\nabla,\tau \in \hat{K}} \chi_\nabla *_K f *_K \chi_\tau \quad \text{where } \chi_\nabla(k) = (\deg \nabla)\operatorname{tr} \nabla(k),$$

and this series converges in $\mathcal{E}(G)$, and for $T \in \mathcal{E}'(G)$ we have

$$\langle T, f\rangle = \sum_{\nabla,\tau \in \hat{K}} \langle T, \chi_\nabla *_K f *_K \chi_\tau\rangle.$$

Next let $0 \neq T \in \mathcal{E}'(G)$ be such that $DT = 0$. As $DT = 0$, we have that $\langle DT, \chi_\nabla *_K f *_K \chi_\tau\rangle = 0$ for all $\nabla, \tau \in \hat{K}$. Fixing $\nabla_0, \tau_0 \in \hat{K}$, we suppose that $\langle T, \chi_{\nabla_0} *_K f *_K \chi_{\tau_0}\rangle \neq 0$. Now define $T_0 \in \mathcal{E}'(G)$ by setting $\langle T_0, f\rangle = \langle T, \chi_{\nabla_0} *_K f *_K \chi_{\tau_0}\rangle$ for $f \in \mathcal{E}(G)$. Now $T_0 \in \mathcal{E}'(G\colon F)$ and $DT_0 = 0$ as $D \in U(\mathfrak{g})^{\mathfrak{k}}$. Thus, $T_0 = 0$, which is a contradiction. Hence, $D\colon \mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective.

3. **Injectivity criterion.** Suppose $D \in U(\mathfrak{g})^{\mathfrak{k}}$. In this section, we give a criterion for the operator $D\colon \mathcal{E}'(G) \to \mathcal{E}'(G)$ to be injective and derive some consequences of this criterion for operators in Z.

LEMMA 3.1 (INJECTIVITY CRITERION). *Suppose $D \in U(\mathfrak{g})^{\mathfrak{k}}$. Then $D\colon \mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective if for no $\omega \in \hat{M}$ is there a finite dimensional space $V \subseteq X(\omega)$ such that $\Pi_{\omega,\nu}(D)\colon V \to V$ and* $\det \Pi_{\omega,\nu}(D)|_V = 0$ *for all ν.*

PROOF. Let $T \in \mathcal{E}'(G\colon F)$ and fix $\omega \in \hat{M}$ such that $\nu \to \Pi_{\omega,\nu}(T) \not\equiv 0$. Now

$$\Pi_{\omega,\nu}(T)\colon X(\omega, T) \to X(\omega, T),$$

and as $D \in U(\mathfrak{g})^{\mathfrak{k}}$, there is a $V \supseteq X(\omega, T)$ such that $\Pi_{\omega\nu}(D)\colon V \to V$. Now as $\nu \to \Pi_{\omega,\nu}(DT) = \Pi_{\omega,\nu}(D)\Pi_{\omega,\nu}(T)$ is analytic and $\det \Pi_{\omega,\nu}(D)|_V \not\equiv 0$, we have that $\nu \to \Pi_{\omega,\nu}(D)\Pi_{\omega,\nu}(T) \not\equiv 0$.

LEMMA 3.2. *If $D \in U(\mathfrak{g})^{\mathfrak{k}}$ satisfies the injectivity criterion of Lemma* 3.1, *then so does D^*, the adjoint of D.*

PROOF. Let $\rho^*: U(\mathfrak{g}) \to U(\mathfrak{g})$ be the linear map such that

$$\rho^*(X_1 \cdots X_r) = (-1)^r X_r \cdots X_1 \qquad (X_i \in \mathfrak{g}_{\mathbf{c}}).$$

Then $D^* = \rho^*(D)$, and our result follows from the fact that, for $u, v \in X(\omega)$, $(\Pi_{\omega,\nu}(D)u, v) = (u, \Pi_{\omega,\bar{\nu}}(D^*)v)$.

Set $R(X)f(g) = (d/dt)f(g \exp tX)|_{t=0}$ and extend R to a representation of $U(\mathfrak{g})$ on $\mathcal{E}(G)$, $\mathcal{D}(G)$, $\mathcal{E}'(G)$ and $\mathcal{D}'(G)$. Then we easily obtain for $\omega \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{c}}^*$ that, for $T \in \mathcal{E}'(G)$, $\Pi_{\omega,\nu}(R(D)T) = \Pi_{\omega,\nu}(T)\,\Pi_{\omega,\nu}(\rho^*(D))$.

We now easily obtain, from Lemma 3.2,

LEMMA 3.3. *If $D \in U(\mathfrak{g})^{\mathfrak{k}}$ satisfies the injectivity criterion of Lemma* 3.1, *the linear map $R(D)$: $\mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective.*

Suppose now that $D \in \mathfrak{Z}$. Then, as is well known, there is a unique $D_0 \in \mathfrak{Z}(\mathfrak{m} \oplus \mathfrak{a})$, the center of $U(\mathfrak{m} \oplus \mathfrak{a})$, such that $D - D_0 \in \mathfrak{n}U(G)$ and the map $D \to D_0$ is an injective ring homomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{m} \oplus \mathfrak{a})$. Now as $D \in \mathfrak{Z}$, the map $\nu \to \Pi_{\omega,\nu}(D)$ is an analytic map into $\mathbf{C}$ and $\Pi_{\omega,\nu}(D) = \nabla_{\omega,\nu}(\rho^*(D_\nu))$. Letting $H_1, \ldots, H_l$ be a basis for $\mathfrak{a}$ we see that

$$D_0 = \sum_{j=1}^{r} \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant l} z(i_1, \ldots, i_j) H_{i_1} \cdots H_{i_j}$$

with each $z(i_1, \ldots, i_j)$ in the center of $U(\mathfrak{m})$. Then

$$\nabla_{\omega,\nu}(\rho^*(D_0)) = \sum_{j=1}^{r} \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant l} (-1)^j \omega(\rho^*(z(i_1, \ldots, i_j)))$$

$$\cdot (\rho + i\nu)(H_j) \cdots (\rho + i\nu)(H_{i_j})$$

and thus we see that D satisfies the injectivity criterion of Lemma 3.1 if and only if for no $\omega \in \hat{M}$ do all $(\rho^*(z(i_1, \ldots, i_j))) = 0$.

The following result is now obvious.

THEOREM 3.1. *If G is a split semisimple Lie group and $0 \neq D \in \mathfrak{Z}$, then D: $\mathcal{E}'(G) \to \mathcal{E}'(G)$ is injective.*

4. **Homogeneous vector bundles on symmetric spaces.** We conclude by applying our techniques to homogeneous vector bundles over G/K.

Let $\tau: K \to Gl(V)$ be an irreducible unitary representation of K and suppose that $\omega = \tau|_M$ is irreducible. (We include the case where τ is trivial.) τ then determines a homogeneous vector bundle $G \times_\tau V$ over G/K and we identify the cross sections of this vector bundle with functions f: $G \to V$ such that:

$$f(gk) = \tau(k)^{-1} f(g) \qquad (g \in G, k \in K).$$

Let $\mathcal{E}(G: \tau)$ $(\mathcal{D}(G: \tau))$, denote the C^∞-cross sections (with compact support). Let $\mathcal{E}'(G: \tau)$ $(\mathcal{D}'(G: \tau))$ be the dual of $\mathcal{E}(G: \tau)$ $(\mathcal{D}(G: \tau))$. Extend $T \in \mathcal{E}'(G: \tau)$ to an operator on all C^∞-functions f: $G \to V$ by setting

$$\langle T,f\rangle = \langle T,f *_K \chi_{\tau^*}\rangle$$

where $\tau^*(k) = (\tau(k)^{-1})^t$, and extend the elements of $\mathcal{D}'(G\colon \tau)$ accordingly. Let $\mathcal{E}(G\colon \tau\colon F)$, $\mathcal{D}(G\colon \tau\colon F)$, $\mathcal{E}'(G\colon \tau\colon F)$, and $\mathcal{D}'(G\colon \tau\colon F)$ denote, respectively, the elements of $\mathcal{E}(G\colon \tau)$, $\mathcal{D}(G\colon \tau)$, $\mathcal{E}'(G\colon \tau)$, and $\mathcal{D}'(G\colon \tau)$ which are left K-finite.

If $D \in U(\mathfrak{g})^{\mathfrak{k}}$ we see that $R(D)$ maps each of the above eight spaces into itself. In this section, we show that if $R(D) \not\equiv 0$ on $\mathcal{E}'(G\colon\tau)$, $R(D)\colon \mathcal{E}'(G\colon \tau) \to \mathcal{E}'(G\colon \tau)$ is injective.

Observe that we have the following.

Lemma 4.1. *If $f \in \mathcal{E}(G\colon \tau)$, $f *_K \chi_{\tau^*} = f$.*

Lemma 4.2. *If $f \in \mathcal{D}(G\colon \tau)$ and $l \in V^*$, let $l(f)\colon G \to \mathbf{C}$ be the function $l(f(x)) = (l,f(x))$. Then for $g \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{c}}^*$, $\Pi_{\phi,\nu}(l(f)) = 0$ unless $\phi = \omega^*$. Moreover, if $u \in X(\omega^*)$, $\nabla \in \hat{K}$ and $\chi_\nabla *_K u = u$, then $\Pi_{\omega^*,\nu}(l(f))u = 0$ unless $\nabla = \tau^*$.*

Proof. Let $u \in H^{\phi,\nu}$. Then

$$(\Pi_{\phi,\nu}(l(f))u)(k) = \int_G (l,f(x))(\Pi_{\phi,\nu}(x)u)(k)\,dx.$$

As $l(f) = l(f) *_K \chi_\tau$, we see that $\Pi_{\phi,\nu}(l(f))u(k) = \Pi_{\phi,\nu}(l(f))(\chi_{\tau^*} *_K u)(k)$. Thus $\Pi_{\phi,\nu}(l(f)) \neq 0$ only if there is a $u \in X(\phi)$ for which $\chi_{\tau^*} *_K u = u$. This yields our result.

Observe by Frobenius reciprocity that the multiplicity of τ^* in $X(\omega^*)$ is 1.

Lemma 4.3. *Let $T \in \mathcal{E}'(G\colon \tau\colon F)$ and $e \in V$. Let $T \otimes e \in \mathcal{E}'(G\colon F)$ be defined by $\langle T \otimes e,f\rangle = \langle T,fe\rangle$. Then for $g \in \hat{M}$ and $\nu \in \mathfrak{a}_c^*$, $\Pi_{\phi,\nu}(T \otimes e) = 0$ unless $\phi = \omega$. Moreover, if $u \in X(\omega)$, $\nabla \in K$ and $X_\nabla *_K u = u$, then*

$$\Pi_{\omega,\nu}(T \otimes e)u = 0$$

unless $\nabla = \tau$.

Proof. Let $f \in \mathcal{D}(G)$ and consider $f_* T$ in place of T. The proof for $f_* T$ is the same as the proof of Lemma 4.2.

Lemma 4.4. *Let $T \in \mathcal{E}'(G\colon \tau\colon F)$ be $\neq 0$, $D \in U(\mathfrak{g})^{\mathfrak{k}}$, and $V^0 = \{f \in X(\omega)\colon \chi_\tau *_K f = f\}$. Then $\Pi_{\omega,\nu}(\rho^*(D))|_{V^0}$ is a scalar operator and $\nu \to \Pi_{\omega,\nu}(\rho^*(D))|_{V^0}$ is a polynomial. Furthermore, $R(D)T = 0$ if and only if $\nu \to \Pi_{\omega,\nu}(\rho^*(D))|_{V^0} \equiv 0$.*

Proof. As $(\tau\colon \omega) = 1$, τ occurs once and only once in $X(\omega)$. Hence, $\Pi_{\omega,\nu}(\rho^*(D))|_{V^0}$, since it commutes with K, is a scalar operator. It is well known that $\nu \to \Pi_{\omega,\nu}(\rho^*(D))|_{V^0}$ is a polynomial map. As $\Pi_{\omega,\nu}(R(D)T) = \Pi_{\omega,\nu}(T)\Pi_{\omega,\nu}(\rho^*(D))$, our result follows from Lemmas 2.2 and 4.3.

Theorem 4.1. *Let $D \in U(\mathfrak{g})^{\mathfrak{k}}$ and suppose that the map $R(D)\colon \mathcal{E}'(G\colon \tau) \to \mathcal{E}'(G\colon \tau)$ is $\not\equiv 0$. Then $R(D)\colon \mathcal{E}'(G\colon \tau) \to \mathcal{E}'(G\colon \tau)$ is injective.*

Proof. If $R(D)T = 0$ for $0 \neq T \in \mathcal{E}'(G\colon \tau)$, then $\nu \to \Pi_{\omega,\nu}(\rho^*(D))|_{V^0} = 0$ and, hence, $R(D)\mathcal{E}'(G\colon \tau) = 0$ by Lemmas 2.2, 4.3 and 4.4. Hence, $R(D)\colon \mathcal{E}'(G\colon \tau) \to \mathcal{E}'(G\colon \tau)$ is injective.

REMARK. For τ trivial this result was proved by S. Helgason.

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