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CONVOLUTIONS OF CONTINUOUS MEASURES AND SETS OF NONSYNTHESIS

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ABSTRACT. Let G be a nondiscrete LCA group, and $A(G)$ the Fourier algebra of G. It is shown that if E is a closed subset of G such that $(\mu * \nu)(E) \neq 0$ for some μ and $\nu \in M_c(G)$, then E is a set of analyticity and contains a set of nonsynthesis for $A(G)$.

Salinger and Varopoulos [5] give a condition for a closed subset of a metrizable LCA group to be a set of analyticity. In this note we shall prove that every set satisfying their condition contains a set of nonsynthesis. This will be done by first improving one of their results in [5] and then applying a theorem of Varopoulos [6]. We shall also give improvements of the main results of Rago [3] and Graham [1].

Let G be a nondiscrete LCA group, and $A(G) = (L^1(\hat{G}))^\wedge$ the Fourier algebra of G. A closed subset E of G is called a set of synthesis if there exists only one closed ideal I in $A(G)$ such that $\bigcap\{f^{-1}(0): f \in I\} = E$. If every function defined on the interval $[-1, 1]$ that operates in $A(E) = A(G)|_E$ can be extended to an analytic function on some neighborhood of $[-1, 1]$ in the complex plane, then E is called a set of analyticity. Finally we let $M(G)$ denote the convolution measure algebra of G, and $M_c(G)$ the ideal of all continuous measures in $M(G)$.

THEOREM 1 (CF. [5]). *If E is a closed subset of G such that $(\mu * \nu)(E) > 0$ for some μ and $\nu \in M_c^+(G)$, then E is a set of analyticity and contains a set of nonsynthesis.*

To prove this, we need the following improvement of Proposition 1 of [5]. Our proof is notationally complicated, but is an easy consequence of repeated applications of the Fubini theorem.

PROPOSITION 1. *Let X and Y be two locally compact (Hausdorff) spaces, and E a Borel subset of the product space $X \times Y$. Suppose that $(\mu \times \nu)(E) > 0$ for some $\mu \in M_c^+(X)$ and $\nu \in M_c^+(Y)$. Then there exist compact (nonempty) perfect sets $K \subset X$ and $L \subset Y$ such that $K \times L \subset E$.*

PROOF. By induction on $n \geqslant 1$, we construct a collection $\{A_{nj}: 1 \leqslant j \leqslant 2^n\}$ of disjoint compact subsets of X and also a collection $\{B_{nk}: 1 \leqslant k \leqslant 2^n\}$ of disjoint compact subsets of Y. When they are defined, we write as follows:

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$$r(n) = 2^n, \qquad E(x) = \{y \in Y: (x,y) \in E\} \quad \text{for } x \in X,$$
$$A(n) = A_{n1} \times A_{n2} \times \cdots \times A_{nr(n)} \subset X^{r(n)},$$
$$\mu_{nj} = \mu|A_{nj} \in M(A_{nj}) \quad \text{for } 1 \leqslant j \leqslant 2^n,$$
$$\mu_n = \mu_{n1} \times \mu_{n2} \times \cdots \times \mu_{nr(n)},$$

and similarly for $B(n) \subset Y^{r(n)}$, $\nu_{nk} \in M(B_{nk})$, $\nu_n \in M(B(n))$. We demand that they satisfy

$$\text{(a)}_n \qquad \int_{A(n)} \left\{ \prod_{k=1}^{r(n)} \nu_{nk} \left[\bigcap_{j=1}^{r(n)} E(x_j) \right] \right\} d\mu_n(x) > 0.$$

Without loss of generality, we may assume that E is compact. Write $A(0) = A_{01} = X$, $\mu_0 = \mu_{01} = \mu$, $B(0) = B_{01} = Y$, $\nu_0 = \nu_{01} = \nu$, and note that

$$\text{(a)}_0 \qquad \int_{A(0)} \nu_{01}(E(x))\, d\mu_0(x) = (\mu \times \nu)(E) > 0$$

by the Fubini theorem and the hypothesis. Suppose that $\{A_{nj}\}_j$ and $\{B_{nk}\}_k$ as above have been defined for some $n \geqslant 0$. In the following arguments, we shall frequently use the Fubini theorem without reference.

Let I_E denote the characteristic function of E. For $x = (x_1, \ldots, x_{r(n)}) \in A(n)$, $y = (y_1, \ldots, y_{r(n)})$ and $y' = (y'_1, \ldots, y'_{r(n)}) \in B(n)$, set

$$(1) \qquad f(x,y,y') = \prod_{j=1}^{r(n)} \prod_{k=1}^{r(n)} I_E(x_j, y_k) \cdot I_E(x_j, y'_k),$$

$z = (y, y')$, and

$$(2) \qquad g(z) = g(y,y') = \int_{A(n)} f(x,y,y')\, d\mu_n(x).$$

Then we have

$$\int_{B(n)^2} g\, d(\nu_n \times \nu_n) = \int_{A(n)} \left\{ \prod_{k=1}^{r(n)} \nu_{nk} \left[\bigcap_{j=1}^{r(n)} E(x_j) \right] \right\}^2 d\mu_n(x),$$

which is positive by (a)_n. Therefore the integral of the function

$$(3) \qquad F(x, x', x'') = \int_{B(n)^2} f(x,z) \cdot f(x',z) \cdot f(x'',z)\, d(\nu_n \times \nu_n)(z)$$

over the set $A(n)^3$ with respect to the measure $\mu_n \times \mu_n \times \mu_n$ is also positive by (2). It follows that there exists a point $x'' = (x_{n1}, x_{n2}, \ldots, x_{nr(n)}) \in A(n)$ such that

$$(4) \qquad \int_{A(n)^2} F(x, x', x'')\, d(\mu_n \times \mu_n)(x, x') > 0.$$

Now we define

$$B'_{nk} = B_{nk} \cap \left[\bigcap_{j=1}^{r(n)} E(x_{nj}) \right], \qquad \nu'_{nk} = \nu | B'_{nk},$$

$$B(n)' = B'_{n1} \times B'_{n2} \times \cdots \times B'_{nr(n)} \subset B(n),$$

$$\nu'_n = \nu'_{n1} \times \nu'_{n2} \times \cdots \times \nu'_{nr(n)} = \nu_n | B(n)'.$$

Then, by (1) and (3), (4) can be written as

$$\int_{A(n)^2} d(\mu_n \times \mu_n)(x, x') \int_{B(n)^2} f(x, z) \cdot f(x', z)\, d(\nu'_n \times \nu'_n)(z) > 0. \tag{5}$$

Since μ is a continuous measure, the set of all $(x, x') \in A(n)^2$ such that $x_j = x'_j$ for some $1 \leqslant j \leqslant r(n)$ has zero $(\mu_n \times \mu_n)$-measure, and a similar assertion holds for $B(n)^2$ and $\nu'_n \times \nu'_n$. It follows from (5) that there exist disjoint compact subsets $A_{(n+1)(2j-1)}$ and $A_{(n+1)2j}$ of A_{nj} $(1 \leqslant j \leqslant r(n))$, and also disjoint compact subsets $B_{(n+1)(2k-1)}$ and $B_{(n+1)2k}$ of B'_{nk} $(1 \leqslant k \leqslant r(n))$ for which $(\mathrm{a})_{n+1}$ holds. (Notice that the integral in (5) with $A(n)^2$ and $B(n)^2$ replaced by $(\prod_{j=1}^{r(n)} A_{(n+1)(2j-1)}) \times (\prod_{j=1}^{r(n)} A_{(n+1)2j})$ and $(\prod_{k=1}^{r(n)} B_{(n+1)(2k-1)}) \times (\prod_{k=1}^{r(n)} B_{(n+1)2k})$, respectively, is the same as the integral in $(\mathrm{a})_{n+1}$.) This completes the induction.

Now we have by construction:

$$x_{nj} \in A_{nj} \qquad (1 \leqslant j \leqslant 2^n), \tag{b}$$

$$\{x_{nj}\}_j \times \left(\cup \left\{ B_{(n+1)k} \colon 1 \leqslant k \leqslant 2^{n+1} \right\} \right) \subset E, \tag{c}$$

$$A_{(n+1)(2j-1)} \cup A_{(n+1)2j} \subset A_{nj} \qquad (1 \leqslant j \leqslant 2^n), \tag{d}$$

$$B_{(n+1)(2k-1)} \cup B_{(n+1)2k} \subset B_{nk} \qquad (1 \leqslant k \leqslant 2^n). \tag{e}$$

We then claim that there exist compact perfect sets $K \subset X$ and $L \subset B$ such that $K \times L \subset E$, where $B = \bigcap_{n=1}^{\infty} (\bigcup_{k=1}^{r(n)} B_{nk})$. To confirm this, let σ_n be the probability measure that assigns mass 2^{-n} to each x_{nj} $(1 \leqslant j \leqslant 2^n)$, and let σ be an arbitrary weak* cluster point of (σ_n) in $M(X)$. Then, by (b) and (d), σ is a continuous probability measure, since the compact sets A_{nj} $(1 \leqslant j \leqslant 2^n)$ are pairwise disjoint. Moreover, we have $(\operatorname{supp} \sigma) \times B \subset E$, since $\{x_{nj}\}_{nj} \times B \subset E$ by (c) and since E is compact. Similarly we can construct a continuous probability measure τ on B. Consequently it suffices to define $K = \operatorname{supp} \sigma$ and $L = \operatorname{supp} \tau$, which completes the proof.

THE PROOF OF THEOREM 1 is now easy. Let E, μ, and ν be as in Theorem 1. Then we have $(\mu \times \nu)(\tilde{E}) = (\mu * \nu)(E) > 0$, where $\tilde{E} = \{(x, y) \in G^2 \colon x + y \in E\}$. It follows from Proposition 1 that there exist compact perfect sets K and L in G such that $K \times L \subset \tilde{E}$, or equivalently, such that $K + L \subset E$. Therefore E is a set of analyticity by Theorem 9.3.5 of [6], and contains a set of nonsynthesis by Theorem 9.2.3 of [6]. (The metrizability and compactness of G in Theorem 9.2.3 of [6] can be easily removed.) This establishes Theorem 1.

For a compact space K, we write $\mathbf{C}_p(K) = \{f \in \mathbf{C}(K) \colon f^p = 1\}$ for $p = 2, 3, \ldots,$ and $\mathbf{C}_\infty(K) = \{f \in C(K) \colon |f| = 1\}$.

COROLLARY 1. *Suppose, in addition to the hypotheses of Theorem* 1, *that G is metrizable. Then there exist two independent Cantor sets K, L in G, and two*

elements p, q of $\{2, 3, \ldots, \infty\}$ having the following properties: (a) *the set $K + L + y_0$ is contained in E for some $y_0 \in G$, and* (b) *given $f \in \mathbf{C}_p(K)$, $g \in \mathbf{C}_q(L)$, and $\varepsilon > 0$, there exists a continuous character γ of G such that $|f - \gamma| < \varepsilon$ on K and $|g - \gamma| < \varepsilon$ on L.*

PROOF. This follows from Proposition 1 and an easy modification of the well-known method of constructing perfect Kronecker (or K_p) sets (see [4, 5.2.4]). We omit the details.

Now, let us say that a subset E of G is algebraically scattered if to each nonzero element y of $Gp(E)$ there exists a countable subset X of G such that $y \not\in Gp(E \cap X^c)$. Notice that if E is such a set, then so is the set $\{nx: n \in \mathbf{Z}, x \in E\}$, where $\mathbf{Z}$ denotes the set of all integers. We define $nE = \{0\}$ for $n = 0$, and $nE = (n-1)E + E$ for $n \geqslant 1$. The following is an improvement of a result of Rago [3] (see also [2]).

THEOREM 2. *Suppose that E is a σ-compact subset of G which is algebraically scattered, and that $\mu_j \in M_c^+(G)$ for $1 \leqslant j \leqslant n+1$. Then we have*

$$\text{(i)} \qquad (\mu_1 * \cdots * \mu_n)[(nE + x) \cap (nE + y)] = 0 \qquad (x \neq y),$$

$$\text{(ii)} \qquad (\mu_1 * \cdots * \mu_n * \mu_{n+1})(nE + y) = 0 \qquad (y \in G).$$

PROOF. If (i) holds, then there exist at most countably many $x \in G$ such that $f(x) \neq 0$, where $f(x) = (\mu_1 * \cdots * \mu_n)(nE + x)$. Therefore the integration of $f(y - x)$ with respect to $d\mu_{n+1}(x)$ yields (ii) since $\mu_{n+1} \in M_c(G)$. In other words, (i) implies (ii).

Defining $\mu_1 * \cdots * \mu_n =$ the unit point measure at $0 \in G$ when $n = 0$, we note that the result is trivial for $n = 0$. So assume that n is a natural number, and that the result holds if n is replaced by $n - 1$. Given $\mu_1, \ldots, \mu_n \in M_c^+(G)$, let μ be their convolution. To confirm (i), we may assume that $x = 0 \neq y$ (if necessary, translate μ_1 by x), and that $nE \cap (nE + y) \neq \varnothing$. Then notice that y is in $nE - nE \subset Gp(E)$. Since E is algebraically scattered, there exists a countable set X in G such that $y \not\in Gp(E \cap X^c)$. Then we have $[n(E \cap X^c)] \cap [n(E \cap X^c) + y] = \varnothing$, so that $nE \cap (nE + y)$ is contained in the union of $(n-1)E + X$ and $(n-1)E + X + y$. Since X is countable, the last two sets have zero μ-measure by the inductive hypothesis. Hence $nE \cap (nE + y)$ itself has zero μ-measure, which completes the proof.

Let E be a subset of G, $p \in \mathbf{Z}$, and n a natural number. We define $p \times E = \{px: x \in E\}$, $E(0) = \{0\}$, and $E(n) = \bigcup(p_1 \times E + \cdots + p_n \times E)$, where the union is taken over all $(p_1, \ldots, p_n) \in \mathbf{Z}^n$. Notice that if $K \subset G$ is a compact set with positive Haar measure, then there exists an absolutely continuous measure $\mu \in M^+(G)$ such that $\mu^n(K) > 0$ for all $n \geqslant 1$. Thus the following theorem is a generalization of a result of Graham [1].

THEOREM 3. *Suppose that $E \subset G$ is an independent set, that $F \subset G$ is an arbitrary set with* Card $F <$ *the cardinality of the continuum, and that $\mu_1, \ldots, \mu_n, \mu_{n+1} \in M_c^+(G)$. Then we have $(\mu_1 * \cdots * \mu_n * \mu_{n+1})(K) = 0$ for every compact set K in $F + E(n)$.*

To prove this, we need the following.

PROPOSITION 2. *Suppose that $E \subset G$ is an independent set, and that*

$X_1, \ldots, X_n$, $X_{n+1} \subset G$ are uncountable sets. Then $\Sigma_{j=1}^{n+1} X_j \not\subset F + E(n)$ for any set $F \subset G$ with Card $F <$ min{Card X_j: $1 \leqslant j \leqslant n+1$}.

PROOF. Suppose by way of contradiction that this is false for some F as above:

$$X_1 + \cdots + X_n + X_{n+1} \subset F + E(n). \tag{1}$$

Without loss of generality, we may assume that n is the least one of all such natural numbers, and that every X_j contains $0 \in G$. Notice that the latter assumption implies $\Sigma_{j \neq k} X_j \subset \Sigma_{j=1}^{n+1} X_j$ for all $1 \leqslant k \leqslant n+1$, and that (1) cannot happen if $n = 0$.

Let $c' = \min\{\text{Card } X_j\colon 1 \leqslant j \leqslant n+1\}$, and let c be the least ordinal such that the set $A = \{1, 2, \ldots, a, a+1, \ldots\}$ of all ordinals less than c has cardinal equal to c'. Let also B be the product space $A \times \{1, 2, \ldots, n+1\}$ with the lexicographic order. We choose an element $w(a,j) \in \Sigma_{i \neq j} X_i$ for each $(a,j) \in B$ as follows. For $(a,j) = (1,1)$, $w(1,1)$ may be an arbitrary element of $\Sigma_{i \neq 1} X_i$. Suppose that the elements $w(a,j) \in \Sigma_{i \neq j} X_i$, $(a,j) < (b,k)$, have been defined for some $(b,k) \in B$. Then we can write by (1)

$$w(a,j) = y(a,j) + \sum_{i=1}^{n} p_i(a,j) x_i(a,j), \tag{2}$$

where $y(a,j) \in F$, $p_i(a,j) \in \mathbf{Z}$, and $x_i(a,j) \in E$ for all $(a,j) < (b,k)$ and $1 \leqslant i \leqslant n$. Let $F(b,k)$ denote $Gp(\{x_i(a,j)\colon 1 \leqslant i \leqslant n, (a,j) < (b,k)\})$, and notice that $\text{Card}(F + F(b,k)) < c'$, since c' is an uncountable cardinal larger than Card F. It follows from the minimality of n that the set $\Sigma_{i \neq k} X_i$ contains an element $w(b,k)$ such that

$$w(b,k) \not\subset F + F(b,k) + E(n-1). \tag{3}$$

This completes our transfinite induction.

Now (3) implies that the elements $x_i(a,j)$ $(1 \leqslant i \leqslant n,\ 1 < a < c,\ 1 \leqslant j \leqslant n+1)$ are different and that

$$p_i(a,j) x_i(a,j) \neq 0 \tag{4}$$

for all $1 \leqslant i \leqslant n$, $1 < a < c$, and $1 \leqslant j \leqslant n+1$. Moreover, there exist $y_j \in F$ $(1 \leqslant j \leqslant n+1)$ such that the set $A_j = \{a \in A\colon a > 1, y(a,j) = y_j\}$ has infinite cardinal larger than Card F, because Card $A = c' >$ Card F. Then $a_1 \in A_1, \ldots, a_{n+1} \in A_{n+1}$ imply

$$\sum_{j=1}^{n+1} \left\{ y_j + \sum_{i=1}^{n} p_i(a_j,j) x_i(a_j,j) \right\} = \sum_{j=1}^{n+1} w(a_j,j) \tag{5}$$

by (2), and the sum in the right-hand side of (5) belongs to the set $\Sigma_{j=1}^{n+1} \Sigma_{i \neq j} X_i = n(\Sigma_{j=1}^{n+1} X_j)$. Since the last set is contained in $n(F + E(n)) = nF + E(n^2)$ by (1), it follows that there exists $y = y(a_1, \ldots, a_{n+1}) \in nF$ such that

$$y_0 + \sum_{j=1}^{n+1} \sum_{i=1}^{n} p_i(a_j,j) x_i(a_j,j) \in y + E(n^2), \tag{6}$$

where $y_0 = \Sigma_{j=1}^{n+1} y_j$. Since each A_j has infinite cardinal larger than Card F, there must be distinct elements $a_1, b_1, \ldots, a_{n+1}, b_{n+1}$ in A such that $a_j, b_j \in A_j$ for all j and such that $y(a_1, \ldots, a_{n+1}) = y(b_1, \ldots, b_{n+1})$. But then we

have by (6)

$$\sum_j \left\{ \sum_i p_i(a_j, j) x_i(a_j, j) - \sum_i p_i(b_j, j) x_i(b_j, j) \right\} \in E(2n^2),$$

which contradicts the independence of E by (4) (notice that the elements $x_i(a, j)$, $a > 1$, are distinct).

To prove Theorem 3, we need the fact that Proposition 1 holds for more than two factors: if $X_1, \ldots, X_n$ are n locally compact spaces, and if E is a Borel subset of their product space such that $(\mu_1 \times \cdots \times \mu_n)(E) > 0$ for some $\mu_1 \in M_c^+(X_1), \ldots, \mu_n \in M_c^+(X_n)$, then there exist compact perfect sets $K_j \subset X_j$ $(1 \leqslant j \leqslant n)$ such that $K_1 \times \cdots \times K_n \subset E$. This can be proved along the same lines as Proposition 1. Therefore Theorem 3 follows from Proposition 2 and the well-known fact that every compact perfect set has cardinal larger than or equal to the cardinality of the continuum. We omit the details.

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