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SCALAR CURVATURES ON $O(M)$, $G_2(M)$

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ABSTRACT. We show that every C^∞ f: $G_2(M) \to \mathbf{R}$, M^n a compact connected riemannian manifold $n \geqslant 3$, is the scalar curvature function of some complete riemannian metric on $G_2(M)$, the grassmann bundle of 2 planes over M, except possibly when $K =$ constant $\geqslant 0$. A similar result holds for $O(M)$ bundle of orthonormal frames on M.

This note is an application of Theorem A of [3] and O'Neill's formula for the curvature of a riemannian submersion [5], [4] and [1]. Theorem C of [3] gives an affirmative answer to the question described in the abstract if $f(P) < 0$ for at least one 2-plane section P tangent to M.

Preliminaries. Let (M^n, ds^2) be a compact riemannian manifold and let $O(M)$ be the principal $O(n)$ bundle of ds^2-orthonormal frames on M. Choose a connection of π: $OM \to M$.

We assume the setting of [1, Section 1]. (See also [4].) Briefly, let $\langle\, , \rangle$ be the bi-invariant metric on $O(n)$ defined via the positive definite $-B$ (killing form) on the lie algebra $\hat{G}$ of $O(n)$, and γ the connection form for π. Split each $X \in T(OM)$ into horizontal and vertical components relative to γ: $X = X^H \dotplus X^V$. For $t > 0$, $g_t = \pi^* ds^2 + t^2\langle \gamma, \gamma\rangle$ is a family of $O(n)$-right invariant complete riemannian metrics on P and relative to each one π is a riemannian submersion on (M, ds^2). Fix a $t > 0$ and let $g \equiv g_t$. Let U be open in M and $X'_1, \ldots, X'_n$ be a ds^2-orthonormal (o.n. from now on) frame on U. Consider reductive decomposition of $\hat{G} = \hat{N} \dotplus \hat{H}$, orthogonal relative to $-B$ where $\hat{H}$ is the lie algebra of $O(2) \times O(n-2)$, and $\hat{N}$ is the space of skew symmetric matrices of the form:

$$\begin{pmatrix} 0 & 0 & -\xi^t \\ 0 & 0 & -\eta^t \\ \xi & \eta & 0 \end{pmatrix}, \quad \xi,\ \eta \text{ column vectors in } R^{n-2},$$

(see [2, vol. II, p. 280]).

Let e_1^r, e_2^s, $r, s = 1, \ldots, n-2$, be the obvious $(-B)$-o.n. basis of $\hat{N}$, where

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e_1^r is the matrix we obtain above if ξ is the rth standard basis element of R^{n-2} and $\eta = 0$. Similarly with e_2^s. Let $\hat{H} = \hat{H}_1 \dot{+} \hat{H}_{n-2}$ with $\hat{H}_1 = \{(\begin{smallmatrix} 0 & -r \\ r & 0 \end{smallmatrix}), r \in R\}$ the lie algebra of $O(2)$ and $\hat{H}_{n-2}$ the lie algebra of $O(n-2)$.

Let $X'_{n+1}, \ldots, X'_{3n-4}; X'_{3n-3}; X'_{3n-2}, \ldots, X'_{n+r}$ be notation for an o.n. basis of $\hat{N} \dot{+} \hat{H}_1 \dot{+} \hat{H}_2$, where $r = \dim O(n)$. If $A \to A^*$ denotes the lie algebra monomorphism $\hat{G} \to \mathfrak{X}(OM)$, the vector fields of $O(M)$, determined by the free $O(n)$ action, we have that $X_1, \ldots, X_n; t^{-1}X^*_{n+1}, \ldots, t^{-1}X^*_{n+r}$ form a local g-o.n. basis on $\pi^{-1}(U)$, where X_i is the γ-horizontal lift of X'_i.

Consider now the following

$$\begin{array}{ccccc} O(M), & g \xrightarrow{p} & \dfrac{O(M)}{O(2) \times O(n-2)} & \equiv G_2(M), \hat{g} \\ \pi \downarrow & & \downarrow q & \\ M, ds^2 & = & M, ds^2 & \end{array}$$

where $\hat{g}$ is the metric on $G_2(M)$ with respect to which p is a riemannian submersion. Let $u \in \pi^{-1}(U) \subset O(M)$, $p(u) \in G_2(M)$, $x = \pi(u) \in M$. We want to calculate $S_{\hat{g}}(p(u))$: the scalar curvature at $p(u)$ relative to $\hat{g}$.

A $\hat{g}$-o.n. frame at $p(u)$ is obtained by projecting a g-o.n., p-horizontal (i.e., normal to the p-fibre relative to g) frame at u of $O(M)$. From $qp = \pi$ follows that Horiz (p) = Horiz $(\pi) \dot{+}$ span $\{X_{n+1}, \ldots, X_{3n-4}\}$, $X_{n+s} = t^{-1}X^*_{n+s}$ (sum orthogonal rel g). From now on let $1 \leqslant \alpha, \beta \leqslant n+r$, $3n-3 \leqslant a \leqslant n+r$, $n+1 \leqslant b, \eta, \theta \leqslant n+r$, $n+1 \leqslant \lambda, \mu \leqslant 3n-4$, $1 \leqslant i, j, k \leqslant n$.

Let $\bar{K}_{\alpha\beta} \equiv K_{\hat{g}}(p_*(X_\alpha), p_*(X_\beta))$ where X_α is one of the X_i's or $t^{-1}X^*_\lambda$'s and $K_{\hat{g}}$ is the sectional curvature relative to $\hat{g}$. By O'Neill's formula for the curvature of a riemannian submersion ([5], [4], [1]),

$$\bar{K}_{\alpha\beta} = K_g(X_\alpha, X_\beta) + \tfrac{3}{4}\|[X_\alpha, X_\beta]^V\|^2, \tag{1}$$

where V stands for "vertical part" or "g-orthogonal projection onto the p-fibre" in any $T_u(OM)$. Notice that

$$\begin{aligned} [X_i, X_j]^V &= \sum_a g([X_i, X_j], X_a) X_a = \sum_a t^2 \langle \gamma[X_i, X_j], \gamma(X_a) \rangle X_a \\ &= \sum_a \langle \gamma[X_i, X_j], X'_a \rangle X^*_a. \end{aligned}$$

If $\| \ \|$ is the length relative to g,

$$\|[X_i, X_j]^V\|^2 = t^2 \sum_a \langle \gamma_{ij}, X'_a \rangle^2,$$

where $\gamma_{ij} = \gamma[X_i, X_j] \in \hat{G}$.

Let $\langle \gamma_{ij}, X'_a \rangle(u) \equiv 2H^a_{ij}(u)$ and obtain

$$\bar{K}_{ij} = K_{ij} + 3t^2 \sum_a (H^a_{ij})^2 \tag{2}$$

Similarly, $\bar{K}_{i,\lambda} \equiv K_{\hat{g}}(p_* X_i, p_* X_\lambda) = K(X_i, X_\lambda) + \frac{3}{4}\|[X_i, X_\lambda]^V\|^2$. Here X_i is π-

horizontal and X_λ is a fundamental p-vertical. $\therefore$ $[X_i, X_\lambda]$ is zero.

$$\overline{K}_{i\lambda} = K(X_i, X_\lambda). \tag{3}$$

Now, $\overline{K}_{\lambda,\mu} = K(X_\lambda, X_\mu) + \frac{3}{4}\|[X_\lambda, X_\mu]^V\|^2$. Recall that $X_\lambda = t^{-1}X^*_\lambda$ and the basis X'_λ was exactly the e_1^r's and e_2^s's above. It is $[e_1^r, e_1^s] = [e_2^r, e_2^s] = 0$ and

$$[e_1^r, e_2^s] = \delta_{rs}\begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ & 0 \end{pmatrix} \quad \text{in } \hat{H}_1 \subset \hat{G}.$$

If $-B$ is so normalized as to have e_1^r, e_2^s of length one then $[e_1^r, e_2^s]$ is also of length 1.

$$[X_\lambda, X_\mu] = [t^{-1}X^*_\lambda, t^{-1}X^*_\mu] = t^{-2}[X^*_\lambda, X^*_\mu].$$

From the above discussion $\|[X_\lambda, X_\mu]^V\|^2 = (t^{-2})^2 t^2 1$ or it is zero depending on λ, μ. The number of nonzero ones is $n - 2$.

From this and (2), (3) we have:

$$\begin{aligned} \sum_{\alpha,\beta} \overline{K}_{\alpha,\beta} &= \sum_{i<j} \overline{K}_{ij} + \sum_{i,\lambda} \overline{K}_{i\lambda} + \sum_{\lambda<\mu} \overline{K}_{\lambda,\mu} \\ &= \sum_{i<j} \left\{ K_{ij} + 3t^2 \sum_a (H^a_{ij})^2 \right\} + \sum_{i,\lambda} K_{i\lambda} \\ &\quad + \sum_{\lambda<\mu} K_{\lambda\mu} + \tfrac{3}{4}t^{-2}(n-2). \end{aligned} \tag{4}$$

Now from O'Neill's theorem [5] on the submersion π one can express K_{ij}, $K_{i\lambda}$, $K_{\lambda\mu}$ relative to the curvature of M and the π-fibre $O(n)$.

$$K_{ij} \equiv K(X_i, X_j) = K_{ds^2}(X'_i, X'_j) - \tfrac{3}{4}\|[X_i, X_j]^W\|^2$$

where W denotes the vertical component relative to the π-fibre:

$$\begin{aligned} [X_i, X_j]^W &= \sum_b g([X_i, X_j], X_b)X_b \\ &= \sum_b \langle \gamma[X_i, X_j], X'_b \rangle X^*_b = 2\sum_b H^b_{ij} X^*_b, \end{aligned}$$

$$\|[X_i, X_j]^W\|^2 = 4t^2 \sum_b (H^b_{ij})^2, \qquad (H^b_{ij} \equiv H^b_{ij}(u)).$$

$$K_{ij} = K_{ds^2}(X'_i, X'_j) - 3t^2 \sum_b (H^b_{ij})^2. \tag{5}$$

$K_{i\lambda} = t^2\|(\nabla_{X_i} X_\lambda)^{\text{hor}}\|^2$ ([5], [4]) where ∇ is the riemannian connection corresponding to g, and hor stands for π-horizontal component. It is easy to see from [5, Lemma 2, p. 446] or [4], that

$$\|(\nabla_{X_i} X_\lambda)^{\text{hor}}\|^2 = \sum_j \tfrac{1}{2} g([X_i, X_j], X_\lambda)^2$$

$$= \tfrac{1}{4} \sum_j (2H_{ij}^\lambda)^2, \tag{6}$$

$$K_{i\lambda} = t^2 \sum_j (H_{ij}^\lambda)^2.$$

Note. This is the same result as in Proposition 5 of [1] where $K_{i\lambda} \equiv R_{i\lambda i\lambda}$. $K_{\lambda\mu} = K(X_\lambda, X_\mu)$ and it must be the curvature of the same section considered as tangent to the totally geodesic π-fibre, i.e.,

$$K_{\lambda\mu} = \tfrac{1}{4} t^{-2} \|[X_\lambda^*, X_\mu^*]\|^2$$

where $\| \;\|$ is the original $O(n)$ norm. (This agrees with Proposition 5 of [1]:

$$K_{\lambda\mu} = R_{\lambda\mu\lambda\mu} = \tfrac{1}{4} t^{-2} \sum_f (C_{\mu f}^\lambda)^2,$$

where $C_{st}^r = \langle [X_s, X_t], X_r \rangle = C_{rt}^s$, etc.) But $\|[X_\lambda^*, X_\mu^*]\| = 1$ or 0 as above and there are exactly $(n-2)$ combinations $(\lambda < \mu)$ that give us 1:

$$\sum_{\lambda<\mu} K_{\lambda\mu} = \tfrac{1}{4} t^{-2} (n-2). \tag{7}$$

From (4), (5), (6), (7) we have:

$$\sum_{\alpha,\beta} \overline{K}_{\alpha\beta} = \sum_{i<j} K_{ds^2}(X_i', X_j') - \sum_{i<j} 3t^2 \sum_b (H_{ij}^b)^2$$
$$+ \sum_{i<j} 3t^2 \sum_a (H_{ij}^a)^2 + \sum_{i,\lambda} t^2 \sum_j (H_{ij}^\lambda)^2$$
$$+ \tfrac{1}{4} t^{-2}(n-2) + \tfrac{3}{4} t^{-2} (n-2).$$

Therefore,

$$S_{\hat{g}}(p(u)) = S_{ds^2}(\pi(u)) - 2t^2 \sum_\lambda \sum_{i<j} (H_{ij}^\lambda(u))^2 + t^{-2}(n-2)$$

after collecting terms and observing the ranges of the indices a, b, and λ. ($S_{ds^2}(\pi(u))$ is the scalar curvature of ds^2 on M at $\pi(u)$.)

$$\Lambda(U) \equiv 2 \sum_\lambda \sum_{i<j} (H_{ij}^\lambda(u))^2 \geqslant 0 \quad \text{for all } u \in \pi^{-1}(U).$$

Let S be the notation for the scalar curvature of ds^2 on M, $S\colon M \to R$ and $\Lambda\colon O(M) \to R$ but it factors through M by the $O(n)$-invariance of g. So, instead of $\Lambda(u)$ we write $\Lambda(x)$, $x = \pi(u)$.

We proved:

PROPOSITION. *The scalar curvature of $\hat{g}$ at $p(u)$ is equal to $S(x) - t^2\Lambda(x) + t^{-2}(n-2)$ where $x = \pi(u)$, and $\Lambda(x) \geqslant 0$.*

THEOREM. *A C^∞ function f: $G_2(M) \to R$ is a scalar curvature function for some riemannian metric on $G_2(M)$ except perhaps when it is a nonnegative constant.*

The proof of this theorem is an application of

THEOREM A ([3]). *Let (N, g) be a smooth compact riemannian manifold with gaussian (resp. scalar if* dim $N \geqslant 3$) *curvature S and let $f \in C^\infty(N)$. If there is a constant $c > 0$ such that*

$$\min cf < S(x) < \max cf$$

for all $x \in N$, then there is a smooth metric g_1 on N with gaussian (resp. scalar) curvature f.

Recall that $2H_{ij}^{\lambda} = \langle \gamma[X_i, X_j], X'_\lambda \rangle$ and X_i, X_j are γ-horizontal vectors in $T(OM)$ of unit length relative to $g \equiv g_t$ and therefore of unit length relative to every g_t, $t > 0$. The length of $\gamma[X_i, X_j]$ in $\hat{G}$ is bounded independent of t, i.e., $0 \leqslant \Lambda_m \leqslant \Lambda(x) \leqslant \Lambda_M$ for all $x \in M$ with Λ_m, Λ_M constants independent of t.

Since $S(x)$ is bounded, it follows that $\hat{S}_m(t)/\hat{S}_M(t) \to 1$ as $t \to 0$, where $\hat{S}_m(t)$ and $\hat{S}_M(t)$ are the maximum and minimum of the scalar curvature of $\hat{g}_t$, and the proof is complete.

Special case. If $f =$ constant > 0 is the sectional curvature of (M, ds^2) then $M = S^n/\Gamma$, $\Gamma \subset O(n+1)$ and $O(M) = O(n+1)/\Gamma$, where Γ is a finite subgroup of $O(n+1)$ [6, p. 69]. If M is 1-connected, $M = S^n$ of radius $f^{-1/2}$ and therefore $G_2(M) = O(n+1)/O(2) \times O(n-2)$, which admits a constant scalar curvature function as a homogeneous space of $O(n+1)$. Any C^∞ function from $G_2(M) \to R$ will then be a scalar curvature for some riemannian metric on $G_2(M)$, by Theorem C of [3].

The following is proved exactly the same way as the above theorem.

PROPOSITION. *A C^∞ f: $O(M) \to R$ is a scalar curvature function for some riemannian metric on $O(M)$, except perhaps when $f =$ constant $\geqslant 0$.*

PROOF. Using the same notation conventions,

$$K_{ij} = K_{ds^2}(X'_i, X'_j) - 3t^2 \sum_\eta (H_{ij}^\eta)^2,$$

$$K_{i\theta} = t^2 \sum_j (H_{ij}^\theta)^2,$$

$$K_{\eta\theta} = \tfrac{1}{4} t^{-2} \langle [X_\eta^*, X_\theta^*], [X_\eta^*, X_\theta^*] \rangle = t^{-2} K_0(X'_\eta, X'_\theta)$$

where K_0 is the sectional curvature of the fibre $O(n)$ in its original metric $\langle \ , \ \rangle$. Therefore, $S_g(u) = S(x) - t^2\Lambda_1(u) + t^{-2}c$ where $S_g(u)$ is the scalar curvature of $O(M)$ relative to the metric $g \equiv g_t$ at u,

$$\Lambda_1(u) = 2 \sum_{i<j} \sum_\eta (H_{ij}^\eta(u))^2 \geqslant 0$$

bounded independent of t and c is the constant positive scalar curvature of $(O(m), \langle\ ,\ \rangle)$.

If $0 \leqslant f_m < f_M$ by the exact same procedure as in the theorem we obtain that f is a scalar curvature on $O(M)$.

In the particular case that f is $K \circ p$ with K the constant positive curvature of M, then $M = S^n/\Gamma$ and $O(M) = O(n+1)/\Gamma$ is a homogeneous space of a compact lie group that admits a metric with positive constant scalar curvature. By Theorem C of [3], all C^∞ functions on $O(M)$ are scalar curvatures and in particular f.

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