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A NOTE ON THE HUREWICZ THEOREM IN SHAPE THEORY

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ABSTRACT. In this note the following is proved: For a pointed movable continuum, if the first shape group of it is a countable group, then the first Hurewicz theorem in shape theory holds. However, in general, it does not hold without the assumption of countability.

The notion of shape was originally introduced by Borsuk. In this note we shall use the ANR-systems approach to shape theory of Mardešić and Segal [4]. Concerning the Hurewicz theorem in shape theory there are many papers (see [3], [5]–[8]). There are two such styles of the theorem. One of them is a description in the category of pro-groups (see [5]–[7]). The other is a description in the category of groups (see [3], [5] and [8]). However, in all previous papers 1-shape connectedness is assumed; hence in this paper we shall prove the following

THEOREM. *Let X be a compact connected metric space and x be a point of X such that (X, x) is pointed movable. Then the limit Hurewicz homomorphism $h\colon \check{\pi}_1(X,x) \to \check{H}_1(X)$ is onto, and if the kernel of h is a countable group, then it coincides with the commutator subgroup of $\check{\pi}_1(X,x)$ where $\check{\pi}_1$ and $\check{H}_1$ are the first shape group and the first Čech homology group functor, respectively. Furthermore, the assumption of countability is essential.*

PROOF. Since X is a compact connected metric space, there exists an inverse sequence $\{(X_i, x_i), \rho_i^{i+1}\}$ such that $\varprojlim\{(X_i, x_i)\} = (X, x)$ and each X_i is a connected polyhedron (see [4]). Then we have two inverse sequences of groups, $\{\pi_1(X_i, x_i), \pi_1(\rho_i^{i+1})\}$ and $\{H_1(X_i), H_1(\rho_i^{i+1})\}$, where π_1 and H_1 are the first homotopy group functor and the first homology group functor, respectively. Since each X_i is a connected polyhedron, we have the Hurewicz homomorphism $h_i\colon \pi_1(X_i, x_i) \to H_1(X_i)$ such that $h_i \pi_1(\rho_i^{i+1}) = H_1(\rho_i^{i+1}) h_{i+1}$ for each i. By the classical Hurewicz theorem, each h_i is onto and the kernel of h_i is $\langle \pi_1(X_i, x_i)\rangle$, where $\langle G\rangle$ is the commutator subgroup of a group G. Hence we obtain the following exact sequence:

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$$* \to \{\langle \pi_1(X_i, x_i)\rangle\} \to \{\pi_1(X_i, x_i)\} \to \{H_1(X_i)\} \to * .$$

By taking inverse limits we obtain the map

$$h = \varprojlim\{h_i\}\colon \check{\pi}_1(X, x) = \varprojlim\{\pi_1(X_i, x_i)\} \to \check{H}_1(X) = \varprojlim\{H_1(X_i)\}$$

which is the *limit Hurewicz homomorphism*, and $\check{\pi}_1(X,x)$, $\check{H}_1(X)$ are the first shape group of (X,x) and the first Čech homology group of X, respectively. By using the above exact sequence and movability of (X,x), Mardešić and Ungar [5, Theorem 8] proved the following.

Claim 1. *h is onto and the kernel of h is* $\varprojlim\{\langle \pi_1(X_i, x_i)\rangle\}$.

Next we investigate some relations between $\langle \check{\pi}_1(X,x)\rangle$ and $\varprojlim\{\langle \pi_1(X_i, x_i)\rangle\}$. Let $\rho_i\colon \varprojlim\{\langle \pi_1(X_i, x_i)\rangle\} \to \langle \pi_1(X_i, x_i)\rangle$ be the projection and K_i the image of ρ_i for each i. Hence $\{K_i\}$ forms an inverse sequence such that $\varprojlim\{K_i\} = \varprojlim\{\langle \pi_1(X_i, x_i)\rangle\}$. From the definition of the commutator subgroup it is obvious that $\langle \check{\pi}_1(X,x)\rangle \subset \varprojlim\{\langle \pi_1(X_i, x_i)\rangle\}$. Let us define $\tau_i\colon \langle \check{\pi}_1(X,x)\rangle \to K_i$ by $\tau_i = \rho_i | \langle \check{\pi}_1(X,x)\rangle$.

Claim 2. *Each τ_i is onto.*

PROOF. Take an element $\alpha \in K_i$. By the definition of K_i, there exists $\tilde{\alpha} \in \varprojlim\{\langle \pi_1(X_i, x_i)\rangle\}$ such that $\rho_i(\tilde{\alpha}) = \alpha$. Since (X,x) is movable, then (X,x) is uniformly movable (see [9]). Hence there exists $j, j > i$ and $t^{jk}\colon (X_j, x_j) \to (X_k, x_k)$ such that

$$(1) \qquad \rho_k^{k+1} t^{j,k+1} \simeq t^{jk} \quad \text{and} \quad \rho_i^k t^{jk} \simeq \rho_i^j \quad \text{for each } k \geqslant j.$$

Let $\rho_j(\tilde{\alpha}) = \alpha_*$. Since $\alpha_* \in K_j \subset \langle \pi_1(X_j, x_j)\rangle$, there exist finite elements $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in \pi_1(X_j, x_j)$ such that

$$\alpha_* = \langle a_1, b_1\rangle\langle a_2, b_2\rangle \cdots \langle a_m, b_m\rangle$$

where $\langle a, b\rangle = aba^{-1}b^{-1}$ for each $a, b \in \pi_1(X_j, x_j)$. Let us define $a_i^k \in \pi_1(X_k, x_k)$ as follows:

$$a_i^k = \begin{cases} \pi_1(t^{jk})(a_i) & \text{for } k \geqslant j, \\ \pi_1(\rho_k^j)\pi_1(t^{jj})(a_i) & \text{for } k \leqslant j, \end{cases}$$

where we recall that π_1 is the first homotopy group functor. Similarly we can define $b_i^k \in \pi_1(X_k, x_k)$. Let us define $\hat{a}_i$ and $\hat{b}_i$ by $(a_i^k)_{k=1,2,\ldots}$ and $(b_i^k)_{k=1,2,\ldots}$, respectively. Hence by (1), $\hat{a}_i$ and $\hat{b}_i$ are elements of

$$\varprojlim\{\pi_1(X_i, x_i)\}.$$

Put $\beta = \langle \hat{a}_1, \hat{b}_1\rangle\langle \hat{a}_2, \hat{b}_2\rangle \cdots \langle \hat{a}_m, \hat{b}_m\rangle$. It follows from (1) that $\tau_i(\beta) = \alpha$. This completes the proof of Claim 2.

Next, let us consider the kernel of τ_i which is denoted by $\operatorname{Ker}\tau_i$. We have the following exact sequence by Claim 2:

$$* \to \{\operatorname{Ker}\tau_i\} \to \{\langle \check{\pi}_1(X,x)\rangle\} \xrightarrow{\{\tau_i\}} \{K_i\} \to * .$$

Hence by Proposition 2.3 of [1, p. 252], we obtain the following exact sequence:

$$(2)\qquad \begin{aligned} &* \to \varprojlim\{\operatorname{Ker}\tau_i\} \to \varprojlim\langle\check{\pi}_1(X,x)\rangle \xrightarrow{\tau} \varprojlim\{K_i\} \\ &\xrightarrow{\delta} \varprojlim{}^1\{\operatorname{Ker}\tau_i\} \to \varprojlim{}^1\{\langle\check{\pi}_1(X,x)\rangle\} \to \varprojlim{}^1\{K_i\} \to * \end{aligned}$$

where $\tau = \varprojlim\{\tau_i\}$.

By the definition of K_i and τ_i, $\varprojlim\{K_i\} = \varprojlim\{\langle\pi_1(X_i,x_i)\rangle\}$ and

$$\tau\colon \varprojlim\{\langle\check{\pi}_1(X,x)\rangle\} = \langle\check{\pi}_1(X,x)\rangle \to \varprojlim\{\langle\pi_1(X_i,x_i)\rangle\}$$

is the inclusion homomorphism. Hence $\operatorname{Ker}\tau = \varprojlim\{\operatorname{Ker}\tau_i\} = *$.

Since each bonding map of $\{\langle\check{\pi}_1(X,x)\rangle\}$ is the identity map, then $\{\langle\check{\pi}_1(X,x)\rangle\}$ satisfies the Mittag-Leffler condition (see [1, p. 256] or [5, Definition 4] for the Mittag-Leffler condition). Hence, by Corollary 3.5 of [1, p. 256],

$$\varprojlim{}^1\{\langle\check{\pi}_1(X,x)\rangle\} = * .$$

Therefore we have the following exact sequence from (2):

$$(3)\qquad * \to \langle\check{\pi}_1(X,x)\rangle \xrightarrow{\tau} \varprojlim\{\langle\pi_1(X_i,x_i)\rangle\} \xrightarrow{\delta} \varprojlim{}^1\{\operatorname{Ker}\tau_i\} \to * .$$

Hence we have the following.

Claim 3. $\langle\check{\pi}_1(X,x)\rangle = \varprojlim\{\langle\pi_1(X_i,x_i)\rangle\}$ *if and only if* $\varprojlim{}^1\{\operatorname{Ker}\tau_i\} = *$.

Claim 4. *If* $\langle\check{\pi}_1(X,x)\rangle$ *is a countable group then* $|\varprojlim{}^1\{\operatorname{Ker}\tau_i\}| = 1$ *or* $> \aleph_0$ *where* $|A|$ *denotes the cardinality of a set* A.

PROOF. This claim is pointed out by Gray [2] when each $\operatorname{Ker}\tau_i$ is an abelian group. However we have to consider it for nonabelian groups. Note that $\langle\check{\pi}_1(X,x)\rangle \supset \operatorname{Ker}\tau_1 \supset \operatorname{Ker}\tau_2 \supset \cdots$. First we suppose that there exists i_0 such that $\operatorname{Ker}\tau_{i_0} = \operatorname{Ker}\tau_j$ for each $j \geqslant i_0$. Then

$$* = \varprojlim\{\operatorname{Ker}\tau_i\} = \bigcap_{i=1}^{\infty} \operatorname{Ker}\tau_i = \operatorname{Ker}\tau_{i_0} = \operatorname{Ker}\tau_j$$

for each $j \geqslant i_0$. Hence by Claim 2 each τ_j is an isomorphism for $j \geqslant i_0$. Therefore $\tau = \varprojlim\{\tau_i\}$ is an isomorphism. Hence by (3), $|\varprojlim{}^1\{\operatorname{Ker}\tau_i\}| = 1$. Next we can consider the following case: For each i, $\operatorname{Ker}\tau_{i+1} \neq \operatorname{Ker}\tau_i$. Since $K_i = \langle\check{\pi}_1(X,x)\rangle/\operatorname{Ker}\tau_i$ by Claim 2, we have that

$$\varprojlim\langle\pi_1(X_i,x_i)\rangle = \varprojlim\{K_i\} = \varprojlim\{\langle\check{\pi}_1(X,x)\rangle/\operatorname{Ker}\tau_i\}.$$

Since $\operatorname{Ker}\tau_{i+1} \neq \operatorname{Ker}\tau_i$ for each i, $|\varprojlim\{\langle\check{\pi}_1(X,x)\rangle/\operatorname{Ker}\tau_i\}| > \aleph_0$ by the proof of the lemma of Gray [2]. Furthermore, it is easy to prove that

$$|\varprojlim\{\langle\pi_1(X_i,x_i)\rangle\}/\tau\langle\check{\pi}_1(X,x)\rangle| = |\varprojlim{}^1\{\operatorname{Ker}\tau_i\}|$$

where $\lim\{\langle\pi_1(X_i,x_i)\rangle\}/\tau\langle\check{\pi}_1(X,x)\rangle$ means the left coset of $\varprojlim\{\langle\pi_1(X_i,x_i)\rangle\}$ by the subgroup $\tau\langle\check{\pi}_1(X,x)\rangle$ and this bijection is induced by δ. Since $|\langle\check{\pi}_1(X,x)\rangle|$

$\leqslant \aleph_0$, it is easy to prove that $|\varprojlim^1\{\operatorname{Ker}\tau_i\}| > \aleph_0$. This completes the proof of Claim 4.

Now by combining Claim 1, Claim 4 and (3), it is easy to prove the following.

Claim 5. *If* Ker h *is a countable group then* Ker h *coincides with the commutator subgroup of* $\check{\pi}_1(X, x)$.

Next, we have to show that the assumption of countability is essential. For it we construct the following example.

Claim 6. *There exists a compact connected metric space X and a point x of X such that (X, x) is pointed movable and the kernel of the limit Hurewicz homomorphism does not coincide with $\langle\check{\pi}_1(X, x)\rangle$.*

PROOF. First we need some groups. That is, there exists a finite group G_i for each $i = 1, 2, \ldots$ such that $\langle\prod_{i=1}^{\infty} G_i\rangle \neq \prod_{i=1}^{\infty}\langle G_i\rangle$. These groups were obtained by E. Witt [10, p. 128, §2]. Since each G_i is a finite group, there exists a 2-dimensional connected finite CW complex X_i and a point x_i of X_i such that $\pi_1(X_i, x_i) = G_i$ for each i by the well-known homotopy realization theorem. Let us define X and x by $\prod_{i=1}^{\infty} X_i$ and $\prod_{i=1}^{\infty} x_i$, respectively. It is easy to prove that (X, x) is pointed movable, and that $\check{\pi}_1(X, x) = \prod_{i=1}^{\infty} G_i$, and the kernel of the limit Hurewicz homomorphism is $\prod_{i=1}^{\infty}\langle G_i\rangle$. Hence (X, x) has the required properties.

Now, our Theorem is completely proved by Claims 1, 5 and 6.

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