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MORE ON THE "ZERO-TWO" LAW

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ABSTRACT. The Ornstein-Sucheston "zero-two" law for Markov operators is extended, and its proof simplified.

Let (X, Σ, m) be a measure space with $m(X) = 1$. Let P be a linear operator on $L_\infty(X, \Sigma, m)$ satisfying

(a) $P1 = 1$.

(b) *If* $f \geqslant 0$ *a.e. then* $Pf \geqslant 0$ *a.e.*

The operator P is a Markov operator if it satisfies in addition

(c) *If* $f_n \downarrow 0$ *a.e. then* $Pf_n \to 0$ *a.e.*

The operator P is called ergodic and conservative if, in addition to (a), (b) and (c), it satisfies:

(d) *If* $0 \leqslant f$ *a.e. and* $Pf \leqslant f$ *a.e. then* $f =$ const. *a.e.*

See [1] for discussion of these properties.

Let P_1 and P_2 satisfy (a) and (b). Define, as in [1, p. 54], their minimum $P_1 \wedge P_2$ by:

If $0 \leqslant f \in L_\infty$ *then*

$$(P_1 \wedge P_2)f = \inf\{P_1 g + P_2(f - g) | 0 \leqslant g \leqslant f\}.$$

Then $P_1 \wedge P_2$ *is again a linear operator on* L_∞ *such that* $(P_1 \wedge P_2)1 \leqslant 1$ *and* $(P_1 \wedge P_2)f \geqslant 0$ *a.e. whenever* $f \geqslant 0$ *a.e.*

Now

$$\begin{aligned}(P_1 \wedge P_2)1 &= \inf\{P_1 g + P_2(1 - g) | 0 \leqslant g \leqslant 1\}\\ &= 1 - \sup\{P_2 g - P_1 g | 0 \leqslant g \leqslant 1\}.\end{aligned}$$

Put $f = 2g - 1$: $0 \leqslant g \leqslant 1$ if and only if $-1 \leqslant f \leqslant 1$. Thus

$$(*) \qquad (P_1 \wedge P_2)1 = 1 - \tfrac{1}{2}\sup\{P_2 f - P_1 f | -1 \leqslant f \leqslant 1\}.$$

Note that $P_1 \wedge P_2$ satisfy (c) if either P_1 or P_2 does.

Let k be a fixed integer and put $S_n = P^n \wedge P^{n+k}$. Now $P^{n+k} = S_n + R_n' = S_n P^k + R_n'' = S_n(I + P^k)/2 + R_n$ where R_n', R_n'' and $R_n = \frac{1}{2}(R_n' + R_n'')$ are positive operators. Thus

$$\begin{aligned}P^{n_1+n_2+2k} &= S_{n_1} P^{n_2+k}(I + P^k)/2 + R_{n_1} P^{n_2+k}\\ &= S_{n_1} S_{n_2}(I + P^k)^2/4 + R_{n_1,n_2}, \qquad R_{n_1,n_2} \geqslant 0.\end{aligned}$$

Repeat this argument r times to conclude:

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$$P^{n_1+n_2+\cdots+n_r+rk} = S_{n_1}S_{n_2}\cdots S_{n_r}(I+P^k)^r/2^r + R_{n_1,n_2,\ldots,n_r}, \qquad R_{n_1,n_2,\ldots,n_r} \geqslant 0. \tag{1}$$

Let $n_1, n_2, \ldots, n_r$ be chosen and put $m = n_1 + \cdots + n_r + rk$; $T_1 = S_{n_1}\cdots S_{n_r}$; $U = R_{n_1,\ldots,n_r}$, then:

$$\begin{aligned} P^{2m} &= T_1P^m(I+P^k)^r/2^r + UP^m \\ &= T_1P^m(I+P^k)^r/2^r + UT_1(I+P^k)^r/2^r + U^2 \\ &= T_2(I+P^k)^r/2^r + U^2, \qquad T_2 \geqslant 0,\ U \geqslant 0. \end{aligned}$$

Repeat this argument j times to conclude:

$$P^{jm} = T_j(I+P^k)^r/2^r + U^j, \qquad T_j \geqslant 0,\ U \geqslant 0. \tag{2}$$

Finally, as in [3], note that

$$\begin{aligned} &\frac{1}{2^r}\left\|T_j(I+P^k)^r(I-P^k)\right\| \\ &\qquad \leqslant \frac{1}{2^r}\sum_{s=0}^{r-1}\left|\binom{r}{s} - \binom{r}{s+1}\right| + \frac{1}{2^{r-1}} \leqslant \frac{\text{const.}}{\sqrt{r}}. \end{aligned} \tag{3}$$

(The last estimate follows from the fact that $\binom{r}{s}$ increases for $0 \leqslant s \leqslant r/2$ and decreases for $r/2 \leqslant s \leqslant r$, it suffices to consider even integers, r.)

THEOREM I. *Let P satisfy* (a) *and* (b). *If* $\|P^{n_0+k} - P^{n_0}\| < 2$, *for some integer n_0, then* $\lim_{n\to\infty}\|P^{n+k} - P^n\| = 0$.

PROOF. Note that

$$\begin{aligned} S_{n_0}1 &= 1 - \tfrac{1}{2}\sup\{P^{n_0+k}f - P^{n_0}f \mid -1 \leqslant f \leqslant 1\} \\ &\geqslant 1 - \tfrac{1}{2}\|P^{n_0+k} - P^{n_0}\| = \alpha > 0. \end{aligned}$$

Choose $n_1 = n_2 = \cdots = n_r = n_0$ then, by (1) and (2),

$$\|U\| = \|U1\| = \|1 - S_{n_0}^r 1\| \leqslant 1 - \alpha^n = \beta < 1.$$

Thus by (2) and (3),

$$\|P^{jm}(I-P^k)\| \leqslant (\text{const.}/\sqrt{r}) + \beta^j.$$

Since $\|P^n(I-P^k)\|$ decreases with n, the theorem follows.

Our next result is the Ornstein-Sucheston "zero-two" law (see [3]).

THEOREM II. *Let P^n satisfy* (a), (b), (c), *and* (d) *for every n. Then either*

$$\sup\{P^{n+k}f - P^nf \mid -1 \leqslant f \leqslant 1\} = 2 \ a.e.$$

or

$$\lim_{n\to\infty}\left(\sup\{P^{n+k}f - P^nf \mid -1 \leqslant f \leqslant 1\}\right) = 0 \ a.e.$$

PROOF. As in [2], define

$$h_n = \sup\{P^{n+k}f - P^nf \mid -1 \leqslant f \leqslant 1\}$$

then

$$S_n 1 = 1 - \frac{h_n}{2}, \qquad 0 \leq h_n \leq 2,\ h_n \geq h_{n+1} \text{ and } Ph_n \geq h_{n+1}.$$

Thus $h_n \downarrow h$ where $Ph \geq h$. By (d) $h = \text{const.} = \alpha$. Now if $\alpha < 2$ then we may find integers $n_1, \ldots, n_r$ with $S_{n_1} S_{n_2} \ldots S_{n_r} 1 \not\equiv 0$:

$$S_{n_1} S_{n_2} \ldots S_{n_r} S_n 1 \uparrow \left(1 - \frac{\alpha}{2}\right) S_{n_1} S_{n_2} \ldots S_{n_r} 1 \not\equiv 0.$$

(We used (c) here.) Thus in equation (2), $U1 = R_{n_1, n_2, \ldots, n_r} 1 \not\equiv 1$. Let $g = \lim U^j 1$ (note that $1 \geq U1 \geq U^2 1 \geq \cdots$) then $g = Ug \leq P^m g$. Since P^m satisfies (d), $g = \text{const.}$ but then $S_{n_1} \ldots S_{n_r} g = 0$ and by our choice of $n_1, \ldots, n_r$ we must have $g = 0$. Finally, by (3)

$$\left|P^{jm}(I - P^k) f\right| \leq \frac{\text{const.}}{\sqrt{r}} + 2R^j 1.$$

Since the right-hand side tends to zero, independently of f, $-1 \leq f \leq 1$, and h_n is a monotone sequence, we must have $h_n \to 0$ a.e.

REMARK. Let us follow [3] and note that if P is in the 0-class and $u \in L_1$ then $\|u(P^{n+k} - P^n)\| = \langle u, (P^{n+k} - P^n) f\rangle$ for some $-1 \leq f \leq 1$ thus $\|u(P^{n+k} - P^n)\| \underset{n\to\infty}{\to} 0$. Now the closure of $L_1(I - P^k)$ is

$$V = \left\{V \in L_1 \langle v, g\rangle = 0 \quad \text{for all } g \in L_\infty, g = P^k g\right\}$$

by the Hahn Banach Theorem. Thus $\|vP^n\| \to 0$ whenever $v \in V$. By (d) applied to P^k, $v \in V$ if and only if $v \in L_1$ and $\int v\, dm = 0$.

If P is induced by an invertible measure preserving transformation then $\|uP^n\| = \|u\|$ and P is necessarily in the 2-class.

If P is in the 2-class and the transition probabilities $P^n(x, \cdot)$ are defined, then

$$2 = \sup\left\{P^{n+k} f - P^n f \mid -1 \leq f \leq 1\right\} \leq \left\|P^{n+k}(x,\cdot) - P^n(x,\cdot)\right\| \leq 2.$$

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