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ON THE RADON-NIKODÝM THEOREM AND LOCALLY CONVEX SPACES WITH THE RADON-NIKODÝM PROPERTY

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ABSTRACT. Let F be a quasi-complete locally convex space, (Ω, Σ, μ) a complete probability space, and $L^1(\mu; F)$ the space of all strongly integrable functions $f: \Omega \to F$ with the Egoroff property. If F is a Banach space, then the Radon-Nikodým theorem was proved by Rieffel. This result extends to Fréchet spaces. If F is dual nuclear, then the Lebesgue-Nikodým theorem for the strong integral has been established. However, for nonmetrizable, or nondual nuclear spaces, the Radon-Nikodým theorem is not available in general. It is shown in this article that the Radon-Nikodým theorem for the strong integral can be established for quasi-complete locally convex spaces F having the following property:

(CM) For every bounded subset $B \subset l_N^1\{F\}$, the space of absolutely summable sequences, there exists an absolutely convex compact metrizable subset $M \subset F$ such that $\Sigma_{i=1}^{\infty} p_M(x_i) < 1$, $\forall (x_i) \in B$.

In fact, these spaces have the Radon-Nikodým property, and they include the Montel (DF)-spaces, the strong duals of metrizable Montel spaces, the strong duals of metrizable Schwartz spaces, and the precompact duals of separable metrizable spaces. When F is dual nuclear, the Radon-Nikodým theorem reduces to the Lebesgue-Nikodým theorem. An application to probability theory is considered.

0. **Introduction**. The primary purpose of this article is to establish the Radon-Nikodým theorem for a class of quasi-complete locally convex spaces F having the following property:

(CM) For every bounded subset $B \subset l_N^1\{F\}$, the space of absolutely summable sequences, there exists an absolutely convex compact metrizable subset $M \subset F$ such that $\Sigma_{i=1}^{\infty} p_M(x_i) < 1$, $\forall (x_i) \in B$.

The consideration of this problem is motivated by the desire to prove the existence of generalized conditional expectations in the theory of random distributions. Since the random elements of a random distribution are continuous linear functionals, it will not be unreasonable that the Radon-Nikodým theorem can be established, in most cases under consideration, in the duals of certain locally convex spaces.

In [3], [12], the Radon-Nikodým theorem of Rieffel [18] was generalized to Fréchet spaces. The Lebesgue-Nikodým theorem for certain nuclear and dual

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nuclear spaces was obtained by various authors [2], [12], [21], [22]. Furthermore, different versions of the Radon-Nikodým theorem for locally convex spaces were presented by various authors, see for instance [12], [15], [18], [19], [24]. Nevertheless, the analogue of Rieffel's theorem was considered only recently in [10] for normed spaces, and in [3] for Fréchet spaces. It will be shown that this theorem can be established for quasi-complete locally convex spaces having property (CM) above. These spaces include the Montel (DF)-spaces, the strong duals of metrizable Montel spaces, the strong duals of metrizable Schwartz spaces, and the precompact duals of separable metrizable spaces.

The basic idea used here lies in the construction of a Banach space over the range of the vector measure in question, and in the subsequent application of Rieffel's Radon-Nikodým theorem. In §1, some preliminary results are given. In §2 various examples of locally convex spaces having property (CM) are exhibited. In §3, the Radon-Nikodým theorem is proved, and finally in §4, an application of the Radon-Nikodým theorem is considered.

1. **Preliminaries.** In the sequel, only standard terminologies in the theory of locally convex spaces (l.c.s.'s for short) will be used, and only the notations of [20] will be adopted.

Let (Ω, Σ, μ) be a fixed probability space, where Ω is an abstract set, Σ a σ-algebra of subsets of Ω, and μ a probability measure defined on Σ. Without loss of generality, one can consider Σ to be μ-complete. Let $\Sigma^+ \equiv \{S \in \Sigma | \mu(S) > 0\}$.

Let F be a l.c.s. with the 0-neighborhood base $\mathfrak{U}(F)$ (or simply $\mathfrak{U}$ if no confusion can arise). For every $U \in \mathfrak{U}$, let p_U denote the associated continuous seminorm. Let m: $\Sigma \to F$ be a vector measure. For every $U \in \mathfrak{U}$, and $S \in \Sigma$, the *U-variation of m over S* is defined to be

$$V(m, U)(S) \equiv \sup\left\{ \sum_{i=1}^{n} p_U(m(S_i)) | S \supset S_i \in \Sigma, S_i \text{ disjoint}, 1 \leqslant i \leqslant n \right\}.$$

$V(m, U)(\cdot)$ is an extended real-valued measure. m is said to have *bounded variation* iff $V(m, U)(\Omega) < \infty$, $\forall U \in \mathfrak{U}$. m is *μ-continuous*, denoted by $m \ll \mu$ iff $\forall U \in \mathfrak{U}$, $V(m, U)(\cdot) \ll \mu$. Let $S \in \Sigma^+$, then the *average range of m over S* is the set

$$A_S(m) \equiv \{ m(T)/\mu(T) | S \supset T \in \Sigma^+ \}.$$

A function ϕ: $\Omega \to F$ is called *simple* iff ϕ is of the form $\phi(\omega) = \sum_{i=1}^{m} x_i I_{S_i}(\omega)$, $\forall \omega \in \Omega$, where $\{S_i\}_{i=1}^{m} \subset \Sigma$ are disjoint, I_{S_i} is the indicator function of S_i, and $\{x_i\}_{i=1}^{m} \subset F$. A function f: $\Omega \to F$ is *Borel measurable* iff $f^{-1}(V) \in \Sigma$, $\forall$ open $V \subset F$. Simple functions are Borel measurable. However, a function which is the a.e. (μ) limit of a sequence of simple functions need *not* be Borel measurable in general, unless F is metrizable.

DEFINITION 1.1. A *Borel* measurable function f: $\Sigma \to F$ is said to have the *Egoroff property* if there exists a sequence of simple functions, ϕ_n, such that $\lim_n \phi_n = f$ a.e. (μ), and $\forall S \in \Sigma^+$, $\varepsilon > 0$, there exists $T \subset S$, $T \in \Sigma^+$, such that $\mu(S \setminus T) < \varepsilon$, and $\lim_n \phi_n = f$ uniformly on T.

Let $\mathfrak{M}(\mu; F) \equiv \{f\colon \Omega \to F \mid f$ is Borel measurable with the Egoroff property$\}$. $\mathfrak{M}(\mu; F)$ is a vector space. The following lemma follows directly from Definition 1.1.

LEMMA 1.1. *Let $f \in \mathfrak{M}(\mu; F)$. Then f is separably valued, and $\forall\, S \in \Sigma^+$, and $\forall\, \varepsilon > 0$, there exists $T \subset S$, $T \in \Sigma^+$ such that $\mu(S \setminus T) < \varepsilon$ and $f(T)$ is precompact.*

DEFINITION 1.2. Let $f \in \mathfrak{M}(\mu; F)$, and $S \in \Sigma$. The *essential range of f over S* is defined to be

$$\mathrm{er}_S(f) \equiv \{x \in F \mid \forall\, U \in \mathfrak{U}, \mu(\{\omega \in S \mid p_U(F(\omega) - x) < 1\}) > 0\}.$$

For $f \in \mathfrak{M}(\mu; F)$, and $U \in \mathfrak{U}$, let $q_U(f) \equiv \int_\Omega p_U(f)\, d\mu$. Let

$$\mathfrak{L}^1(\mu; F) \equiv \{f \in \mathfrak{M}(\mu; F) \mid q_U(f) < \infty, \forall\, U \in \mathfrak{U}\}$$

denote the space of all *strongly integrable* functions, and

$$L^1(\mu; F) \equiv \mathfrak{L}^1(\mu; F)/\eta,$$

where

$$\eta \equiv \{f \in \mathfrak{L}^1(\mu; F) \mid q_U(f) = 0, \forall\, U \in \mathfrak{U}\}.$$

$L^1(\mu; F)$ is a l.c.s. topologized by the family of seminorms $\{q_U \mid U \in \mathfrak{U}\}$. If F is metrizable, then so is $L^1(\mu; F)$. If F is a Banach space, then $\mathfrak{M}(\mu; F)$ is just the space of Bochner measurable functions, and $L^1(\mu; F)$ is the space of Bochner integrable functions. In general the structure and properties of $L^1(\mu; F)$ are not known, and these topics will not be dealt with here. However, it should be pointed out that $L^1(\mu; F)$ contains a fair number of functions. For, let E be a Banach space and $i\colon E \to F$ be a continuous injection. Then for any $g \in L^1(\mu; E)$, $f = i \circ g \in L^1(\mu; F)$.

If F is quasi-complete, then one can show, either by a direct argument or by appealing to the characterization theorem of Chatterji [5], that $f \in L^1(\mu; F)$ is Pettis integrable. Now for every $f \in \mathfrak{L}^1(\mu; F)$, $\mu_f(S) = \int_\Omega f\, d\mu$ defines a vector measure from Σ into F such that μ_f is of bounded variation, and $\forall\, U \in \mathfrak{U}$, $p_U(\mu_f(S)) \leqslant \int_S p_U(f)\, d\mu$ and $V(\mu_f, U)(S) = \int_S p_U(f)\, d\mu$. It follows that if $f, g \in \mathfrak{L}^1(\mu; F)$ such that $f = g$ a.e. (μ), then $\mu_f = \mu_g$.

The proof of the mean value theorem below will be omitted, since the argument is analogous to that contained in [18].

THEOREM 1.1 (MEAN VALUE THEOREM). *Let $f \in \mathfrak{L}^1(\mu; F)$, and $S \in \Sigma^+$. Then $\mu_f(S)/\mu(S) \in \bar{c}[\mathrm{er}_S(f)]$. Hence, $A_S(\mu_f) \subset \bar{c}[\mathrm{er}_S(f)]$.*

2. **Some examples**. In this section, some examples of l.c.s.'s satisfying property (CM) of §0 will be given. For the l.c.s.'s of Example 2.3–2.5, property (CM) will be established by showing that they all possess the following two properties:

(a) Every absolutely convex compact subset of F is metrizable.

(b) For every bounded subset $B \subset l_N^1\{F\}$, the space of absolutely summable sequences, there exists an absolutely convex compact subset $M \subset F$ such that $\sum_{i=1}^{\infty} p_M(x_i) < 1$, $\forall (x_i) \in B$.

REMARK 2.1. Property (B) of Pietsch [16, p. 31] is obtained by replacing the word "compact" in (b) by "bounded and closed". Evidently, the following implications are obvious: (a) and (b) $\Rightarrow$ (CM) $\Rightarrow$ (B), and (b) $\Rightarrow$ (B).

It would be expedient, for the discussion of Examples 2.3–2.5, to give the following two examples of l.c.s.'s having property (a).

EXAMPLE 2.1. Let E be a separable Mackey space, and $F = E'_t$, where $\sigma < t < \beta$, i.e., t is finer than the weak topology but coarser than the strong topology. Then every absolutely convex t-compact subset $M \subset F$ is metrizable. For, t and σ coincide on M; hence M is equicontinuous, since E is Mackey. The metrizability of M follows from the separability of E [20, p. 128, Theorem 1.7]. Thus F has property (a). In general, F does not have property (b). For instance, if E is separable metrizable, and $F = E'_\beta$, then F has property (B) of Pietsch [16, p. 31, Theorem 1.5.8]. Since F is complete, F will have property (b) provided every bounded subset of F is precompact.

EXAMPLE 2.2. Let E be a l.c.s. such that $\forall\, U \in \mathfrak{U}$, $E[U]$ is separable. Then every absolutely convex σ-closed equicontinuous subset of $F = E'_\sigma$ is σ-compact metrizable. For, let $U \in \mathfrak{U}(E)$, then $E[U]$ is a separable normed space [16, p. 82, Proposition 4.4.9]. Let U° be the polar of U with respect to $\langle E, E'\rangle$; then one can identify $(E[U])'$ with $E'[U^\circ]$ [16, p. 15]. Let $M = E'_{U^\circ} \subset E'$, and let M° be the polar of M with respect to $\langle E, E'\rangle$. Then by [20, p. 135, Corollary 1], $\sigma(M, E/M^\circ) = \sigma(E', E)|_M$. Since $E/M^\circ \sim E[U]$, it follows that $\sigma(E'_{U^\circ}, E[U]) = \sigma(E', E)|_M$. Since U° is $\sigma(E'_{U^\circ}, E[U])$-closed equicontinuous, and since by assumption $E[U]$ is separable, so by [20, p. 128, Theorem 128], U° is $\sigma(E'_{U^\circ}, E[U])$-compact metrizable, and hence is $\sigma(E', E)$-compact metrizable. If E is furthermore a Mackey space, and if $F = E'_t$, where $\sigma < t < \beta$, then every absolutely convex t-compact subset of F is metrizable. For, the t-topology and the σ-topology coincide on the t-compact sets, and the fact that E is Mackey implies that such sets are also equicontinuous.

Before proceeding with further examples, some l.c.s.'s with property (B) will be recalled. A l.c.s. is termed *σ-barreled* if every strongly bounded countable subset of E'_β is equicontinuous. Clearly, quasi-barreled spaces are σ-barreled. A σ-barreled l.c.s. with a fundamental sequence of bounded subsets is called dual metric [16, p. 11]. It is known that dual metric spaces have property (B) [16, p. 31, Theorem 1.5.8]. (DF)-spaces [9, p. 396] are dual metric; hence (DF)-spaces all have property (B). In particular, the strong dual of a metrizable l.c.s. has property (B). These spaces will have the stronger property (b), if every bounded subset is precompact.

EXAMPLE 2.3.1. Let F be a metrizable Montel space. Then F has property (CM). Property (a) is clear. Property (b) follows from the fact that F has property (B) and the fact that every closed bounded subset of a Montel space is compact.

EXAMPLE 2.3.2. Let F be a metrizable nuclear space. Then F has property (CM). For, a metrizable l.c.s. is nuclear iff it is dual nuclear [16, p. 78]. Thus F is metrizable dual nuclear. But every closed and bounded subset of a dual nuclear space is compact [16, p. 82], hence F has property (b).

EXAMPLE 2.3.3. Let E be a separable metrizable l.c.s. such that E'_β is co-Schwartz [7, p. 275], [22]. Then $F = E'_\beta$ has property (CM). For, property

(a) follows from Example 2.1, and property (b) follows from the fact that F is a (DF)-space and the fact that every bounded subset of a co-Schwartz space is precompact [22, p. 241].

Since every quasi-complete co-Schwartz space is Montel, and since the converse is not true even when the space under consideration is (DF) [22, p. 242], the next example will be of interest.

EXAMPLE 2.3.4. Let F be a Montel (DF)-space. Then F has property (CM). For, let $E = F'_\beta$; then E is a (FM)-space [9, p. 397] and hence is separable [9, p. 370]. The fact that F has property (a) follows from Example 2.1, since F being Montel is reflexive. F has property (b), since F has property (B) and every bounded subset of F is precompact.

The above examples seem to suggest that a l.c.s. F has property (b) only when F has property (B) and every bounded subset of F is precompact. However, the next example shows that there are l.c.s.'s which have property (b), but not every bounded subset is precompact. Before presenting the next example, a lemma will be stated. The proof of this lemma is standard.

LEMMA 2.1. *Let E be a metrizable l.c.s. Then E'_p, the precompact dual of E, has property* (b).

EXAMPLE 2.4. Let E be a separable metrizable l.c.s. Then $F = E'_p$ has property (CM). For, Example 2.1 shows that F has property (a), since every metrizable l.c.s. is Mackey [19, p. 132]. Lemma 2.1 shows that F has property (b).

This example includes the following spaces.

EXAMPLE 2.4.1. Let F be a (dF)-space of Brauner [1] such that F'_p is separable. Then F has property (CM). For, F'_p is a separable Fréchet space.

EXAMPLE 2.4.2. Let E be a metrizable Montel space, and $F = E'_p$ $(= E'_\beta)$. Then F has property (CM). For, every metrizable Montel space is separable [9, p. 370].

EXAMPLE 2.4.3. Let E be a metrizable Schwartz space. Then $F = E'_p$ $(= E'_\beta)$ has property (CM). For, for every $U \in \mathfrak{U}(E)$, $E[U]$ is separable, since E is a Schwartz space [22, p. 239]. Hence it follows from Example 2.2 that F has property (a), since E is Mackey. Property (b) now follows from Lemma 2.1. This example covers all the nuclear metrizable l.c.s.'s, since every nuclear space is a Schwartz space.

EXAMPLE 2.5. Let F be a quasi-complete dual nuclear l.c.s. Then F has property (CM). For, F, being a quasi-complete dual nuclear space, has property (B) of Pietsch [16, p. 75, Proposition 4.2.9]. Furthermore, in a dual nuclear space, every bounded subset is precompact. Thus F has property (b).

Now every absolutely convex compact subset of F is metrizable. For, let $M \subset F$ be such a set, and let $U = M^\circ$ be the polar of M, U is a 0-neighborhood in F'_β. Since F'_β is nuclear, the normed space, $F'_\beta[U]$, associated with U is separable [16, p. 82, Proposition 4.4.9]. Now since every quasi-complete dual nuclear space is semireflexive [16, p. 83, Proposition 4.4.11], one has

$$\sigma\big((F'_\beta[U])', F'_\beta[U]\big) = \sigma\big((F'_\beta)', F'_\beta\big)|_{(F'_\beta[U])'} = \sigma(E, F')|_{(F'_\beta[U])'}.$$

Thus, if $M \subset F = (F'_\beta)'$ is absolutely convex compact, then $M \subset (F'_\beta[U])'$ is weak*-compact. But $F'_\beta[U]$ is separable, so M is $\sigma((F'_\beta[U])', F'_\beta[U]$-metrizable. The compactness of M implies that on M, the relative topology coincides with $\sigma((F'_\beta[U])', F'_\beta[U])$. Hence, M is metrizable.

This example holds for instance when $F = E'_\beta$, where E is a nuclear barreled space, or when F is a nuclear (F)-space, or a complete nuclear (DF)-space, or sequential projective limits of respectively such spaces.

3. **The Radon-Nikodým theorem.** In this section, the Radon-Nikodým theorem will be established for l.c.s.'s having property (CM) of §0. The idea lies in the construction of a Banach space over the range of the vector measure in question.

Let F be an arbitrary l.c.s., and $M \subset F$ be an absolutely convex compact subset. Let $F_M = \bigcup_{n=1}^{\infty} nM$ and $p_M(x) \equiv \inf\{\alpha > 0 | \alpha x \in M, \ \forall x \in F_M\}$. (F_M, p_M) is a normed space. Let $\{z_i\}_{i=1}^{\infty} \subset F' \subset C(M)$ be dense in F' under the topology of uniform convergence on compact subsets of M. For every $i \geqslant 1$, define $\|z_i\|_M \equiv \sup\{|\langle z_i, w\rangle| \ |w \in M\}$. Since M is bounded, $\|z_i\|_M < \infty$, $\forall\, i \geqslant 1$. Now for every $x \in F_M$, define

$$N(x) \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\langle z_i, x\rangle|}{\|z_i\|M}.$$

Since $\forall x \in M$, $|\langle z_i, x\rangle| \leqslant \|z_i\|_M$, the sum converges $\forall x \in M$. Now M is absorbing in F_M, so $N(x) < \infty$, $\forall x \in F_M$. $N(\cdot)$ is a seminorm on F_M. Furthermore, one has the following lemma. The proof is direct and will be omitted.

LEMMA 3.1. *Let F_M, p_M, and $N(\cdot)$ be as defined above. Then,* (i) *N is a norm on F_M,* (ii) *the topology of N and the relative topology of F coincide on M,* (iii) *$\forall\, U \in \mathfrak{U}(F)$, there exists $\alpha_U > 0$ such that $p_U(x) \leqslant \alpha_U N(x)$, $\forall\, x \in F_M$, and* (iv) *$\forall\, x \in F_M$, $N(x) \leqslant p_M(x)$.*

Now the Radon-Nikodým theorem will be proved. It will be assumed that F is a quasi-complete l.c.s. having property (CM).

THEOREM 3.1 (RADON-NIKODYM THEOREM). *Let (Ω, Σ, μ) be a complete probability space, and m: $\Sigma \to F$ be a vector measure. Then $m = \mu_f$ for some $f \in L^1(\mu; F)$ iff* (i) *$m \ll \mu$, and* (ii) *m has bounded variation.*

PROOF. *Necessity*. Trivial.

Sufficiency. Let m: $\Sigma \to F$ be a vector measure satisfying (i) and (ii). Let $\Lambda \equiv \{(m(S_i))_{i=1}^{\infty} | S_i \in \Sigma, \ S_i \text{ disjoint}\}$. Since m is of bounded variation, $\Lambda \subset l_N^1\{F\}$ is bounded in the π-topology (for, $\forall\, U \in \mathfrak{U}$, $\Sigma_{i=1}^{\infty} p_U(m(S_i)) < V(m, U)(\Omega)$, $\forall\, (m(S_i))_{i=1}^{\infty} \in \Lambda$).

Now F has property (CM), so there exists an absolutely convex compact metrizable subset $M \subset F$ such that $\Sigma_{i=1}^{\infty} p_M(m(S_i)) < 1$, $\forall\, (m(S_i))_{i=1}^{\infty} \in \Lambda$. Let F_M, and p_M be defined as before. Then m: $\Sigma \to (F_M, p_M)$ is a μ-continuous vector measure of bounded variation. In fact, its variation is bounded by 1. However, in general, m need not have locally relatively compact average range in (F_M, p_M).

Now let $N(\cdot)$ be as defined before. In general, (F_M, N) is not complete. Let $(\tilde{F}_M, N)$ be its completion. m: $\Sigma \to (\tilde{F}_M, N)$ is clearly, from (iv) of Lemma

3.1, to be of bounded variation and μ-continuous. Furthermore, m has locally relatively compact average range in $(\tilde{F}_M, N)$. For, m, when considered as a measure with values in (F_M, p_M), has locally bounded average range. Thus, $\forall S \in \Sigma^+$, there exists $T \subset S$, $T \in \Sigma^+$, such that $A_T(m)$ is bounded in (F_M, p_M). This implies that there exists $\gamma > 0$ such that $A_T(m) \subset \gamma M$. It follows then from (ii) of Lemma 3.1 that $A_T(m)$ is relatively compact in $(\tilde{F}_M, N)$. Hence, m has locally relatively compact average range in $(\tilde{F}_M, N)$.

By the Radon-Nikodým theorem for Banach space, there exists a

$$g \in L^1(\mu; (\tilde{F}_M, N))$$

such that $m = \mu_g$. Furthermore, if one let Π be the family of all finite partitions, π, of Ω directed under inclusion, then $\lim_\pi \phi_\pi = g$ in the mean, where $\phi_\pi = \Sigma_{S \in \pi}[m(S)/\mu(S)]I_S$, and there exists a subsequence ϕ_{π_n} such that $\lim_n \phi_{\pi_n} = g$ a.e. (μ) [18, Proposition 1.13], and $\mu_{\phi_{\pi_n}} \to \mu_g$. Observe that ϕ_{π_n}: $\Omega \to \tilde{F}_M \subset F_M$, since $m(S)/\mu(S) \in F_M$. Let i: $(\tilde{F}_M, N) \to F$ be the injection map; then i is continuous by (iii) of Lemma 3.1. Thus, $\lim_n(i \circ \phi_{\pi_n}) = i \circ g$ a.e. (μ). Define $f(\omega) = (i \circ g)(\omega)$. Then, $f(\omega) \in F$ a.e. (μ), is Borel measurable, and has the Egoroff property. From (iii), one has $\int_\Omega p_U(f)\, d\mu < \alpha_U \int_\Omega N(g)\, d\mu < \infty$. Thus $f \in L^1(\mu; F)$, $m = \mu_f$, and $\mu_{(i \circ \phi_\pi n)} \to \mu_f$. Q.E.D.

A quasi-complete l.c.s. F is said to have the Radon-Nikodým property iff for any complete probability space (Ω, Σ, μ) and any m: $\Sigma \to F$, μ-continuous vector measure of bounded variation, there exists an $f \in L^1(\mu; F)$ such that $m = \mu_f$. One deduces easily from Theorem 3.1 the following corollary.

Corollary 3.1. *Let F be a quasi-complete l.c.s. with propeerty* (CM). *Then F has the Radon-Nikodým property.*

It is known that for a Fréchet space (or Banach space), dentability (or s-dentability) and Radon-Nikodým property are equivalent [4], [8]. Thus for a Fréchet space (or Banach space), property (CM) implies dentability.

For certain l.c.s.'s, Theorem 3.1 reduces to the Lebesgue-Nikodým theorem. One obtains theorems of [2, p. 92] from the following.

Corollary 3.2. *Let F be a quasi-complete dual nuclear l.c.s, and m: $\Sigma \to F$ be a vector measure. Then there exists an $f \in L^1(\mu; F)$ such that $m = \mu_f$ iff $m \ll \mu$.*

Proof. It suffices to show that condition (ii) of Theorem 3.1 is satisfied.

If m is not of bounded variation, then there exsits a $U \in \mathcal{U}$, a decreasing sequence $\{S_n\}_{n=1}^\infty \subset \Sigma$, and a sequence of partitions $\{\pi_n\}_{n=1}^\infty$ such that $S_n \in \pi_n$, π_n a partition of S_{n-1}, and for each n,

$$n - \sum_{i \leq n-1} \sum_{S \in \pi_i \setminus \{S_i\}} p_U(m(S)) < \sum_{S \in \pi_n} p_U(m(S)).$$

The family $\Gamma = \bigcup_{n=1}^\infty (\pi_n \setminus \{S_n\})$ is pairwise disjoint, and $(m(S))_{S \in \Gamma}$ is not absolutely summable. But F being dual nuclear implies that every weakly summable sequence is absolutely summable. Thus one arrives at a contradiction. Therefore, m must have bounded variation. Q.E.D.

Remark 3.1. Theorem 1.1 was not used in the proof of Theorem 3.1, since a vector measure that satisfies (i) and (ii) of Theorem 3.1, and has its values in a l.c.s. with property (CM), automatically has locally relatively compact average range. However, Theorem 1.1 is retained, since it will be essential in the proof of the Radon-Nikodým theorem in l.c.s.'s without property (CM).

The above remark leads to

Problem 1. What property must a l.c.s. have in order that the Radon-Nikodým theorem can be established in its full generality?

4. **An application**. Let (Ω, Σ, μ) be a fixed complete probability space, and F be a quasi-complete l.c.s. with the property (CM). A function

$$f \in L^1(\Omega, \Sigma, \mu; F)$$

will be called an integrable *generalized random variable*. Let $\Sigma_0 \subset \Sigma$ be a sub σ-algebra. Then an integrable generalized random variable

$$g \in L^1(\Omega, \Sigma_0, \mu, F)$$

is called a conditional expectation of f relative to Σ_0, if $\int_S g\, d\mu = \int_S f\, d\mu$, $\forall S \in \Sigma_0$. The proof of the next theorem is an immediate consequence of Theorem 3.1.

THEOREM 4.1. *Let $f \in L^1(\Omega, \Sigma, \mu; F)$ be an integrable generalized random variable, and let $\Sigma_0 \subset \Sigma$ be a sub σ-algebra. Then the conditional expectation of f relative to Σ_0 exists. If F is the dual of certain separable l.c.s., then the conditional expectation is unique a.e. (μ).*

COROLLARY 4.1. *Let F be quasi-complete dual nuclear. Then for every $f \in L^1(\Omega, \Sigma, \mu; F)$, there exists, relative to a given sub σ-algebra $\Sigma_0 \subset \Sigma$, a unique (a.e. (μ)) conditional expectation, $E_{\Sigma_0}(f)$.*

PROOF. The uniqueness of the conditional expectation follows from the fact that the normed space $F'_\beta[U]$ of Example 2.5 is separable. Q.E.D.

REMARK 4.1. Let F be a quasi-complete l.c.s. with property (CM). Since an F-valued vector measure that satisfies (i) and (ii) of Theorem 3.1 automatically has locally relatively compact average range, the concept of strong measurability can be weakened. For a discussion of various concepts of measurability, see [2, p. 87].

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