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RANKS OF MATRICES OVER ORE DOMAINS

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ABSTRACT. Let R be a Noetherian Ore domain. Then rank M = inner rank M for every matrix M over R if and only if R is projective-free of global dimension at most 2.

1. Let R be a right and left Ore domain with field of quotients Q and let M be a finitely generated right R-module. The then *rank* $r(M)$ is the Q-dimension of the vector space $M \otimes_R Q$ and we denote by $d(M)$ the least number of elements in a set of generators of M.

If γ is a homomorphism of free R-modules $\gamma: R^n \to R^m$, then the rank $r(\gamma)$ of γ is the rank of the image of γ. The *inner rank* $\rho(\gamma)$ of γ (defined by Bergman [1, p. 126] for arbitrary rings) may be defined to be the minimum of $d(M)$, where $\operatorname{Im}(\gamma) \leqslant M \leqslant R^m$. Alternatively, if G is a matrix for γ, then $\rho(\gamma)$ is the least integer ρ such that $G = G_1 G_2$ with G_1 an $m \times \rho$ and G_2 a $\rho \times n$ matrix. Inner rank and rank do not always coincide, even over commutative domains. In this note we give necessary and sufficient conditions on a Noetherian Ore domain for the two notions of rank to coincide, and thus give a partial answer to a question raised by Bergman [1, p. 150].

2. Throughout, R is a right and left Ore domain with field of quotients Q. All modules are right R-modules, and tensor products are over R.

LEMMA 1. (a) *If* $0 \to N \to R^n$ *is exact then* $N \otimes Q = 0$ *implies that* $N = 0$.
(b) *Let* $0 \to R^n \to M$ *be an exact sequence of R-modules. If* $d(M) \leqslant n$ *then in fact* $M \cong R^n$.

PROOF. Both parts of the lemma are immediate consequences of the exactness of $\otimes_R Q$.

(a) If x is a nonzero element of N then $xR \cong R$. Thus the exactness of $0 \to R \to N$ gives $0 \to Q \to N \otimes Q$ which insures that $N \otimes Q \neq 0$.

(b) Let $0 \to K \to R^n \to M \to 0$ be a presentation for M. Tensoring both sequences with Q, we get the exact diagram

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$$\begin{tikzcd} 0 \arrow[r] & K \otimes Q \arrow[r] & R^n \otimes Q \arrow[r] & M \otimes Q \arrow[r] & 0 \\ & & & R^n \otimes Q \arrow[u] & \\ & & & 0 \arrow[u] & \end{tikzcd}$$

This shows that both maps into $M \otimes Q$ are isomorphisms and hence that $K \otimes Q = 0$. Thus $K = 0$, as needed. $\square$

LEMMA 2. *The following are equivalent*:

(i) *If* $M \leqslant R^n$, *then* $r(M) = n$ *or* $M \leqslant K \leqslant R^n$ *with* $K \simeq R^{n-1}$.

(ii) *If* $M \leqslant R^n$, *then* $r(M) = n$ *or* $M \leqslant K \leqslant R^n$ *with* $d(K) = n - 1$.

(iii) *If* $0 \to K \to R^n \to R$ *is exact, then* $K \simeq R^{n-1}$.

PROOF. (i) $\Rightarrow$ (ii) trivially.

Assume (ii) and let K' be the kernel of a functional $R^n \to R$. Tensoring with Q, we see $r(K') = n - 1$. Thus if $K' \not\simeq R^{n-1}$, then, by Lemma 1, $d(K') \geqslant n$. Then by (ii) $K' \leqslant K \leqslant R^n$ with $d(K) = n - 1$. But K/K', as a nonzero submodule of R, contains a copy of R generated, say, by $k + K'$. But then $kR \cap K' = 0$ so that $K' \oplus kR \leqslant K$ and $r(K) \geqslant n$. This contradicts $d(K) = n - 1$ and so (iii) holds.

Assume (iii) and suppose $M \leqslant R^n$, with $r(M) < n$. Then there is a Q-functional γ: $R^n \otimes Q \to Q$ which vanishes at $M \otimes Q$. Let γ' be the restriction of γ to R^n. Then γ': $R^n \to Q$ is an R-linear map which vanishes at M. Now $\gamma'(R^n)$ is a finitely generated R-module, say $\gamma'(R^n) = q_1 R + \cdots + q_n R$. Since R is also a left Ore domain, there are elements $r, r_1, \ldots, r_n$ in R with $r \neq 0$ and $q_i = r^{-1} r_i$. Thus $r\gamma'(R^n) \subseteq R$. Thus $r\gamma'$ is an R-functional from R^n to R. Since $\gamma'(M) = 0$, also $r\gamma'(M) = 0$. Thus $M \leqslant \operatorname{Ker}(r\gamma')$, which is isomorphic to R^{n-1} by assumption, and (i) holds. $\square$

SOME DEFINITIONS. γ: $R^n \to R^n$ is *full* if $\rho(\gamma) = n$. R has ACC^* if for each n, free R-modules have ACC on n-generator submodules.

PROPOSITION. (a) *Let R satisfy* (i) *and* γ: $R^n \to R^m$. *Then* $\rho(\gamma) = r(\gamma)$.

(b) *Let R have ACC^*. If R does not satisfy* (iii) *then there is a full homomorphism* γ: $R^n \to R^n$ *of rank less than n.*

PROOF. (a) Assume (i) and let γ: $R^n \to R^m$. If $m = 1$ then clearly $\rho(\gamma) = r(\gamma)$ and we use induction on m. If $\rho(\gamma) = m$, then $\operatorname{Im} \gamma$ is not contained in an $m - 1$ generator submodule of R^m. Thus, by (i), $r(\operatorname{Im} \gamma) = m$ and hence $\rho(\gamma) = r(\gamma) = m$. Otherwise $\operatorname{Im} \gamma \leqslant M \leqslant R^m$ with $d(M) = \rho(\gamma) < m$. Thus, by (i), $M \leqslant R^{m-1} \leqslant R^m$. Let γ_1 be the map γ cut down to R^{m-1} and γ_2 be the injection $R^{m-1} \to R^m$. Then $\gamma = \gamma_1 \gamma_2$. Clearly $\rho(\gamma_1) \geqslant \rho(\gamma)$. But since $\operatorname{Im} \gamma_1 \leqslant M \leqslant R^{m-1}$ and $d(M) = \rho(\gamma)$, then $\rho(\gamma_1) = \rho(\gamma)$. Thus $\rho(\gamma_1) = r(\gamma_1)$ by induction. Since γ_2 is one-to-one, $r(\gamma_1) = r(\gamma)$. Thus, finally, $\rho(\gamma) = \rho(\gamma_1) = r(\gamma_1) = r(\gamma)$.

(b) Assume R has ACC* and that γ: $R^n \to R$ is a functional whose kernel K is not isomorphic to R^{n-1}. Since $r(K) = n - 1$, it follows from Lemma 1 that $d(K) \geqslant n$ and also that a free submodule F of K has $r(F) = d(F)$

$\leqslant n-1$. Thus K is not free and, by ACC*, K has the maximal condition on free submodules. Let then $F \leqslant K$ be a submodule of K maximal with respect to being free. Then $r(F) = n-1$. Let x be K but not in F. Let $M = F + xR$. If $d(M) < n$, then M is free, in contradiction with the choice of F. Thus $d(M) = n$. Suppose $M \leqslant T \leqslant R^n$ with $d(T) < n$. Then, since $F \leqslant T$, Lemma 1 insures that $T = R^{n-1}$. Now from the exact sequence $0 \to M \to T \to T/M \to 0$ we get the exact sequence

$$0 \to M \otimes Q \to T \otimes Q \to T/M \otimes Q \to 0$$

which gives $T/M \otimes Q = 0$. Now since $M \leqslant K$ we also have a sequence $T/M \to T + K/K \to 0$ which gives

$$T/M \otimes Q \to T + K/K \otimes Q \to 0.$$

Thus $T + K/K \otimes Q$ is zero and hence by Lemma 1, $T + K/K = 0$, i.e. $T \leqslant K$. This contradicts the maximality of F. It follows that any map $\alpha\colon R^n \to R^n$ whose image is M has inner rank n and rank $n-1$.

The Proposition shows that for Ore domains with ACC, whether rank = inner rank can be decided by considering only full homomorphisms.

If R is a Noetherian (and hence Ore) domain, we can couch the Proposition in homological terms.

THEOREM 1. *Let R be a Noetherian domain. Then inner rank = rank if and only if R has global dimension at most two and finitely generated projective R-modules are free.*

PROOF. By Theorem 21 of [3], gl dim$(R) = 1 +$ hom dim(A) where A is some ideal of R. Present A as $0 \to K \to F \to A \to 0$ where F is a finitely generated free module. If inner rank = rank then (iii) holds so that hom dim $A \leqslant 1$ and hence gl dim$(R) \leqslant 2$. Further, if P is a finitely generated projective with $d = d(P)$ then $P \oplus Q = R^d$ for some Q. If $r(P) = d$ then P is free. If $r(P) < d$, it follows from (i) that $P \leqslant M < R^d$ with $d(M) = d-1$. But P is again a summand of M, so $d(P) \leqslant d-1$, a contradiction. So finitely generated projective R-modules are free. The reverse implication follows in a similar manner.

The Noetherian, or at least the ACC*, hypothesis of Theorem 1 is necessary: considering matrices, let M be an $m \times n$ matrix over R of inner rank ρ. Then $M = M_1 M_2$ where M_1 is $m \times \rho$ and M_2 is $\rho \times n$. Then M has inner rank ρ when considered as a matrix over the ring R' generated by the entries of M_1 and M_2. Also, if R is commutative, $r(M)$ is the rank of M as a matrix over R', since $r(M)$ is the maximal order of a submatrix of M with nonzero determinant. So if R is commutative and inner rank = rank for all finitely generated subrings of R, this is also true for R. Thus a union of (finitely generated) projective-free commutative rings of global dimension $\leqslant 2$ has the property that inner rank = rank. Such a ring may well have global dimension > 2. For example let G be a torsion-free infinitely generated locally cyclic abelian group. Then $\mathbf{Z}G$ has global dimension 3 [2, Theorem 5, p. 149].

3. **Remarks.** (a) David Lissner proved for us that the following is an explicit example of a full 3×3 matrix which has rank 2: let k be a field,

$$R = k[x,y,z], \quad \text{and} \quad A = \begin{pmatrix} -z & 0 & x \\ y & -x & 0 \\ 0 & z & -y \end{pmatrix}.$$

(b) It is easy to see that if every full matrix over $R[x]$ is invertible over $Q(x)$ then every full matrix over R is invertible over Q. Theorem 1 gives a simple proof of the well-known fact that if R is a Dedekind domain and $R[x]$ is projective-free then R is a PID.

(c) Using results of Lissner and Geramita [4, Theorems 2.6 and 3.4], Theorem 1 can be restated in terms of the outer product property: for a commutative Noetherian domain, inner rank = rank if and only if R is an outer product domain which is a UFD.

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