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A GRAPH-THEORETICAL GENERALIZATION OF A CANTOR THEOREM

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ABSTRACT. In 1962 Gleason and Dilworth found a poset-theoretical generalization of the Cantor Theorem concerning the cardinality of power sets. In the present paper we prove a graph-theoretical generalization of both theorems.

0. Dilworth and Gleason have proved in [2] (see also [1, Theorem 4.6]) that no order preserving map from a subset of a partially ordered set P into $I(P)$, the set of all ideals[1] of P, ordered by inclusion, is onto. Their theorem was a generalization of the well-known Cantor Theorem concerning the cardinality of power sets.

Our main purpose is to generalize the former result as follows: if P and Q are two directed graphs and Q is a reflexive nondiscrete graph with at least two elements, then no antihomomorphism from a subgraph of P into $H(P, Q)$, is onto. Here $H(P, Q)$ denotes the graph of all homomorphisms from P into Q considered as a subgraph of Q^P, i.e. with the relation defined coordinatewise. The theorem of Dilworth and Gleason can now be obtained by taking $Q = \{0, 1\}$ with $1 \geqslant 0$, in which case we have that if P is a partially ordered set, then $I(P)$ is anti-isomorphic to $H(P, Q)$.

If Q is a symmetric graph then our theorems yield corresponding theorems for homomorphisms. In general, Theorems 1–4 imply similar results for homomorphisms rather than antihomomorphisms. We shall postpone proofs to a sequel of this paper in which we shall study properties, defined below, C and C* with respect to homomorphisms of arbitrary relational structures.

1. By a relational structure we shall mean here a pair $A = (A, (R_t)_{t \in T})$, where for every $t \in T$, $R_t = R_t(A)$ is a binary relation on the nonempty set A. If A and B are two relational systems of the same type, then the function f: $A \to B$ is said to be a homomorphism if for every $t \in T$ and $(a, b) \in R_t(A)$ we have that $(f(a), f(b)) \in R_t(B)$. If $(a, b) \in R_t(A)$ implies that $(f(b), f(a)) \in R_t(B)$, then f is said to be an antihomomorphism.

If A and B are two systems of the same type, then $H(A, B)$ denotes a relational system, the universe of which is the set of all homomorphisms from

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[1] I.e. subsets $I \subseteq P$ such that $y \leqslant x \in I$, then $y \in I$.

A into B and the relations of which are defined coordinatewise. When we say that B is a subsystem of A we mean that $B \subseteq A$ and for every $t \in T$, we have that $R_t(B) = B^2 \cap R_t(A)$. A system with one relation will be called a graph, $|A|$ denotes the cardinality of the set A. A relational structure is nontrivial if it has more than one element and the relational structure is nondiscrete if there are $t \in T$ and $x \neq y$ such that $(x, y) \in R_t$.

A relational structure Q has property C if for every structure P of the same type as Q there are no antihomomorphisms from P onto $H(P, Q)$. Moreover, Q has property C* if for every substructure P_0 of P there are no antihomomorphisms from P_0 into $H(P, Q)$.

2.

THEOREM 1. *Let Q be a relational system in which each of the relations R_t is reflexive. If Q has exactly two elements then Q has property* C.

PROOF. Let P be a relational structure of the same type as Q and let φ be an antihomomorphism from P into $H(P, Q)$. Let $Q = \{0, 1\}$, and let ϕ: $P \to Q$ be defined as follows: $\phi(x) = 1$ if and only if for every $y \in P$ we have that $\varphi(x)y = 1$ implies that $\varphi(y)y = 0$. We shall show that ϕ is a homomorphism. Suppose that $(a, b) \in R_t(P)$. Because each of the relations $R_t(Q)$ is reflexive, we obviously can assume that $\phi(a) \neq \phi(b)$, say $\phi(a) = 0$ and $\phi(b) = 1$. Because $\phi(a) = 0$, there is $y \in P$, such that $\varphi(a)y = 1$ and $\varphi(y)y = 1$. Because $\varphi(y)y = 1$ and $\phi(b) = 1$ we have that $\varphi(b)y = 0$. Since φ is an antihomomorphism into $H(P, Q)$ and $(a, b) \in R_t$, we must have that for every $y \in P$ the pair $(\varphi(b)y, \varphi(a)y) \in R_t(Q)$. Thus $(0, 1) \in R_t(Q)$, i.e. ϕ is a homomorphism. We shall show now that $\phi \not\in \operatorname{Im} \varphi$. In fact, suppose that $\phi = \varphi(e)$ and $\phi(e) = 1$. Then $\varphi(e)e = 1$ and thus, by the definition of ϕ, we must have that $\varphi(e)e = 0$, which is a contradiction. Suppose that $\phi(e) = 0$. Then there exists y such that $\varphi(e)y = 1$ and $\varphi(y)y = 1$. But $\varphi(e)y = 1$ means that $\phi(y) = 1$, which together with $\varphi(y)y = 1$ implies that $\varphi(y)y = 0$, which is a contradiction. Thus Theorem 1 is proved.

THEOREM 2. *Let Q be a relational structure containing a nontrivial reflexive substructure Q_0 injective in the category of all structures of the given type. If Q_0 has property* C *then Q has property* C*.

PROOF. Let P be a structure of the same type as Q, P_0 a substructure of P and suppose that φ: $P_0 \to H(P, Q)$ is an antihomomorphism onto $H(P, Q)$. Let P_1 be the inverse image of $H(P, Q_0)$ (note that $|H(P, Q_0)| > 1$, because Q_0 is nontrivial and reflexive). Then $\varphi_1 = \varphi | P_1$ maps P_1 onto $H(P, Q_0)$. Let us put $\gamma(h) = h | P_1$ for $h \in H(P, Q_0)$. Since Q_0 is injective, γ is onto; since γ is a homomorphism, $\varphi_2 = \gamma \circ \varphi_1$ is an antihomomorphism. Thus φ_2 is an antihomomorphism from P_1 onto $H(P_1, Q_0)$ which is a contradiction because Q_0 has property C. Thus Q has property C*.

THEOREM 3. *Every nondiscrete reflexive graph has property* C*.

PROOF. Every nondiscrete graph contains a two-element nondiscrete subgraph, and this, by Theorem 1, has property C. On the other hand, it is easy to see that every two-element reflexive graph is injective in the category of all graphs. Thus Theorem 3 follows from Theorem 2.

THEOREM 4. *Every nontrivial reflexive graph Q has property* C.

PROOF. In view of Theorem 3, we can assume that Q is discrete. Let $\alpha(P)$ denote the number of connected components of the graph P. If Q is discrete so is $H(P, Q)$. Because every homomorphism into a discrete graph is constant on connected components and arbitrary otherwise we have

$$\alpha(H(P, Q)) = |H(P, Q)| = |Q|^{\alpha(P)} \geqslant 2^{\alpha(P)} > \alpha(P).$$

Thus there are no antihomomorphisms from P onto $H(P, Q)$.

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