

HYPERINVARIANT SUBSPACES OF REDUCTIVE OPERATORS

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ABSTRACT. T. B. Hoover has shown that if A is a reductive operator, then $A = A_1 \oplus A_2$, where A_1 is normal and all the invariant subspaces of A_2 are hyperinvariant. A new proof is presented of this result, and several corollaries are derived. Among these is the fact that if A is hyperinvariant and T is polynomially compact and $AT = TA$, then $A^*T = TA^*$. It is also shown that every reductive operator is quasitriangular.

1. **Introduction.** A (bounded) operator on a separable Hilbert space is *reductive* if every subspace invariant for the operator also reduces it. We denote by $\{A\}'$ the commutant of A , that is, the set of all operators that commute with A . If \mathcal{Q} is any family of operators, the *lattice* of \mathcal{Q} , denoted $\text{Lat}\mathcal{Q}$, is the set of subspaces invariant under all the operators in \mathcal{Q} . Finally, a subspace \mathfrak{N} is *hyperinvariant* for A if $\mathfrak{N} \in \text{Lat}\{A\}'$.

In [5], T. B. Hoover proved the following theorem:

THEOREM H. *If A is a reductive operator then A can be written as a direct sum $A_1 \oplus A_2$ where A_1 is normal, A_2 is reductive, $\{A\}' = \{A_1\}' \oplus \{A_2\}'$, and all the invariant subspaces of A_2 are hyperinvariant.*

We can state the last part of this theorem as follows: If A is a completely nonnormal reductive operator (that is, A has no normal direct summand) then $\text{Lat}A = \text{Lat}\{A\}'$. The well-known result of Dyer, Pedersen, and Porcelli [2] bears on this subject. That theorem says that every operator has a nontrivial invariant subspace if and only if every reductive operator is normal. It may turn out, therefore, that there are no nonnormal reductive operators and that the restatement of Theorem H deals with an empty class. Until someone proves the invariant subspace conjecture, however, the decomposition of A in Theorem H can be very useful, as I hope §3 of this paper will show.

We refer to a subspace \mathfrak{N} as *hyperreducing* for an operator A if \mathfrak{N} is in $\text{Lat}\{A\}' \cap \text{Lat}\{A^*\}'$, that is, if \mathfrak{N} reduces every operator in $\{A\}'$. Lemma 1 and its corollary were established in [6].

LEMMA 1. *If $A_1 \oplus A_2$ is reductive and $A_1X = XA_2$, then $A_1^*X = XA_2^*$.*

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COROLLARY 1. *If A is reductive, then every hyperinvariant subspace of A is hyperreducing. (Equivalently: $\text{Lat}\{A\}' = \text{Lat}\{A^*\}'$.)*

2. **Theorem H using single-operator techniques.** In [5], the proof of Theorem H depends heavily on the theory of von Neumann algebras, as well as the concept of the invariant algebra developed earlier in that paper. We would like to show in this section that Theorem H can be proved without recourse to von Neumann algebra techniques. Our methods by no means supplant Hoover's, since most of the results in [5] deal with reductive *algebras* and therefore require algebraic techniques. The proof of Theorem 1 which appears below was supplied by the referee.

THEOREM 1. *Let $A_1 \oplus A_2$ be reductive on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and let $A_1X = XA_2$, where $X: \mathcal{H}_2 \rightarrow \mathcal{H}_1$. Then:*

- (1) $(\text{ran } X)^-$ reduces A_1 and $A_1|(\text{ran } X)^-$ is normal.
- (2) $\ker X$ reduces A_2 and $A_2|\ker^\perp X$ is normal.

PROOF. It is obvious that $(\text{ran } X)^-$ reduces A_1 and that $\ker X$ reduces A_2 , since both A_1 and A_2 are reductive. Let \mathfrak{N} be the subspace $\{\langle A_1Xf, f \rangle: f \in \mathcal{H}_2\}$. It is easy to check that \mathfrak{N} is invariant for, and hence reduces, $A_1 \oplus A_2$, so it follows that $(A_1^* \oplus A_2^*)\langle A_1Xf, f \rangle \in \mathfrak{N}$ for all $f \in \mathcal{H}_2$, or $A_1^*A_1X = A_1XA_2^*$. By Lemma 1 we also know that $XA_2^* = A_1^*X$, so the last equation becomes $A_1^*A_1X = A_1A_1^*X$, and it follows that $A_1|(\text{ran } X)^-$ is normal. The second statement of the theorem follows from the first by consideration of adjoints.

PROOF OF THEOREM H. Let \mathfrak{N} be the largest reducing subspace of A such that the restriction of A to \mathfrak{N} is normal. It is easy to see that there is a largest such subspace, since \mathfrak{N} can be characterized as the span of the set $\{\mathfrak{N}: \mathfrak{N} \text{ is a reducing subspace for } A \text{ and } A|_{\mathfrak{N}} \text{ is normal}\}$. Let A_1 and A_2 be the restrictions of A to \mathfrak{N} and \mathfrak{N}^\perp respectively. Then A_1 is normal, and A_2 is reductive and any operator T can be written as a matrix.

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

corresponding to the decomposition of the Hilbert space as $\mathfrak{N} \oplus \mathfrak{N}^\perp$. If T commutes with A then $A_1T_{12} = T_{12}A_2$ and $T_{21}A_1 = A_2T_{21}$. It follows from the first equation and Theorem 1 that $A_2|\ker^\perp T_{12}$ is normal, but since A_2 has no normal direct summand, $\ker^\perp T_{12} = \{0\}$, that is, $T_{12} = 0$. Similarly, $A_2|\text{ran } T_{21}$ is normal and it follows that $T_{21} = 0$.

The fact that every invariant subspace of A_2 is hyperinvariant follows by essentially the same proof, where the direct summands of A_2 play the role of A_1 and A_2 in the preceding paragraph.

Theorem H and Corollary 1 yield the following:

THEOREM H*. *If A is reductive then A can be written as $A_1 \oplus A_2$, where A_1 is normal, A_2 is reductive, $\{A\}' = \{A_1\}' \oplus \{A_2\}'$ and all the invariant subspaces of A_2 are hyperreducing. Equivalently: $\text{Lat}A_2 = \text{Lat}\{A_2\}' \cap \text{Lat}\{A_2^*\}'$.*

The authors of [2] refer to an operator with no normal direct summand as *completely nonnormal*. Let A be a completely nonnormal reductive operator; then the summand A_1 is absent and $\text{Lat}A = \text{Lat}\{A\}' \cap \text{Lat}\{A^*\}'$. Now suppose that T commutes with A and \mathfrak{N} is a hyperinvariant subspace of T . Then \mathfrak{N} is invariant under A and, hence, \mathfrak{N} reduces $\{A\}'$; in particular, \mathfrak{N} reduces T . C. K. Fong [3] has referred to an operator T as *hyporeductive* if every hyperinvariant subspace of T reduces T . Thus we have

COROLLARY 2. *If A is a completely nonnormal reductive operator and $TA = AT$, then T is hyporeductive.*

3. Applications. In [6] it is shown that if A is reductive, C is compact and injective, and $AC = CA$, then A is normal. Using Theorem H we can weaken the hypothesis of injectivity.

THEOREM 2. *Suppose that A is reductive, C is compact and $AC = CA$. Let \mathfrak{N} be the reducing kernel of C , that is, $\mathfrak{N} = \ker C \cap \ker C^*$. Then \mathfrak{N} reduces A and $A|_{\mathfrak{N}^\perp}$ is normal. If $\dim \mathfrak{N} < \infty$, A is normal..*

PROOF. Since $AC = CA$, $\ker C$ reduces A ; since $A^*C^* = C^*A^*$, $\ker C^*$ reduces A . Hence \mathfrak{N} reduces A , and it is clear that \mathfrak{N} reduces C . Let $A' = A|_{\mathfrak{N}^\perp}$ and $C' = C|_{\mathfrak{N}^\perp}$; then $A'C' = C'A'$ and C' has no reducing kernel.

Using Theorem H, write A' as $A_1 \oplus A_2$ where A_2 is completely nonnormal, and C' as $C_1 \oplus C_2$. The kernel of C_2 is a hyperinvariant subspace, and by Corollary 2, $\ker C_2$ reduces C_2 . Since C_2 can have no reducing kernel (C' has none), it must be that C_2 is injective. By the result quoted from [6], A_2 is normal. Since A_2 was chosen to be completely nonnormal, it must be absent, and A' is normal.

If $\dim \mathfrak{N} < \infty$ then $A|_{\mathfrak{N}}$ is a reductive operator acting on a finite-dimensional space and is therefore normal.

In [9], P. Rosenthal introduced the following property which an operator T may have: (P) if \mathcal{A} is any reductive algebra for which $T \in \mathcal{A}'$, then $T^* \in \mathcal{A}'$. Rosenthal showed that if T is n -normal or compact, then T has property (P); further, Radjavi and Rosenthal [7, p. 169] showed the following:

THEOREM 3. *If T is algebraic then T has property (P).*

A weakened version of property (P) would require only that if $T \in \mathcal{A}'$ for a singly generated reductive algebra \mathcal{A} , then $T^* \in \mathcal{A}'$; equivalently, (P') if A is a reductive operator and $TA = AT$, then $T^*A = AT^*$. The results of the previous section seem tailor-made for showing that special types of operators have property (P'). We remark that the full property (P) can be shown for certain operators using results like those in §2, but which deal with reductive algebras, not just single operators.

Theorem 4 generalizes an earlier result of Rosenthal [8] that polynomially compact reductive operators are normal.

THEOREM 4. *If A is reductive, T is polynomially compact, and $AT = TA$, then $A^*T = TA^*$.*

PROOF. By Theorem H and the Fuglede Theorem, and the fact that polynomial compactness is inherited by direct summands, it suffices to prove the theorem in the case that A is completely nonnormal. Let p be a polynomial for which $p(T) = K$, where K is compact. Then A also commutes with K . If \mathfrak{N} is the reducing kernel of K then $A|_{\mathfrak{N}^\perp}$ is normal (Theorem 2), a fact which contradicts the complete nonnormality of A unless $\mathfrak{N}^\perp = \{0\}$, that is, $K = 0$ and T is algebraic. Theorem 3 now applies and the proof is complete.

C. K. Fong has also proved Theorem 4 in the general case where T is polynomially compact and $T \in \mathcal{Q}'$ for \mathcal{Q} a reductive algebra [4].

THEOREM 5. *If A is completely nonnormal and reductive, and if T commutes with A , then T is quasitriangular.*

Notice that this theorem is not true for all reductive operators— for instance, the identity.

PROOF. If T^* has an eigenvalue λ , let \mathfrak{N}_λ be the kernel of $T^* - \lambda$. \mathfrak{N}_λ is hyperinvariant for T^* and thus reduces T by Corollary 2. Now suppose that T is nonquasitriangular and let \mathfrak{N} be the span of all the eigenvectors of T^* . The subspace \mathfrak{N} reduces T and $T|_{\mathfrak{N}}$ is diagonal, so it must be that $T|_{\mathfrak{N}^\perp}$ is nonquasitriangular. But then $T^*|_{\mathfrak{N}^\perp}$ would have an eigenvector [1, Theorem 5.5], which contradicts the choice of \mathfrak{N} . Thus T must be quasitriangular.

COROLLARY 3. *Every reductive operator is quasitriangular.*

PROOF. Let A be reductive and use Theorem H to write A as $A_1 \oplus A_2$. The operator A_1 is normal (hence quasitriangular), and A_2 commutes with itself and is therefore quasitriangular by Theorem 5.

Corollary 3 can also be proved without recourse to Theorem H.

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